Direct measurement of the $Q$-function in a lossy cavity

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Abstract

We show that the $Q$-function corresponding to an electromagnetic field in a lossy cavity can be directly measured by means of a simple scheme, therefore allowing the knowledge of the state of the field despite dissipation.

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The reconstruction of a quantum state is a central topic in quantum optics and related fields [1,2]. During the past years, several techniques have been developed, for instance the direct sampling of the density matrix of a signal mode in multiport optical homodyne tomography [3], tomographic reconstruction by unbalanced homodyning [4], reconstruction via photocounting [5], cascaded homodyning [6] to cite some. There have also been proposals to measure electromagnetic fields inside cavities [7,8] and vibrational states in ion traps [7,12]. In fact, the full reconstruction of nonclassical states of the electromagnetic field [13] and of (motional) states of an ion [14] have been experimentally accomplished. The quantum state reconstruction in cavities is usually achieved through a finite set of selective measurements of atomic states [7] that make it possible to construct quasiprobability distribution functions such as the Wigner function, that constitute an alternative representation of a quantum state of the field.

Nevertheless, in real experiments, the presence of noise and dissipation has normally destructive effects. Schemes that treat a lossy cavity have been proposed [8] that involve a physical process that allows the storage of information about the quantum coherences of the initial state in the diagonal elements of the density matrix of a transformed state. The importance of determining states of the quantized field are of great importance, in particular, given recent experimental achievements on the generation of Fock states in superconducting microwave cavities [9]. The method proposed here would represent one possible state-determination technique in the presence of dissipation. The relation between losses and $s$-parametrized quasiprobability distributions has already been pointed out.
in [10] and problems with the reconstruction of the Wigner function have been analyzed in [11]. Here, we would like to retake the problem of reconstruction of the cavity field as studied in [7], but allowing the cavity to have losses (i.e., a real cavity). We then will show, that although it is not possible to reconstruct the Wigner function, it is still possible to recover whole information about the initial state via the Q-function.

Let us first consider the ideal case of no dissipation. We consider then the Hamiltonian for the interaction between a quantized field and a two-level atom reads (we have set \( \hbar = 1 \))

\[
\hat{H} = \omega a^\dagger \hat{a} + \frac{\omega_{eg}}{2}\hat{\sigma}_z + \lambda (\hat{a}^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \hat{a}),
\]

(1)

where \( \hat{a} \) and \( \hat{a}^\dagger \) are the creation and annihilation operators for the field mode, respectively, obeying \([\hat{a}, \hat{a}^\dagger] = 1 \). \( \hat{\sigma}_\pm = |e\rangle\langle g| \) and \( \hat{\sigma}_z = |g\rangle\langle e| \) are the raising and lowering atomic operators, respectively, \(|e\rangle \) being the excited state and \(|g\rangle \) the ground state of the two-level atom. The atomic operators obey the commutation relation \([\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z \). \( \omega \) is the field frequency, \( \omega_{eg} \) the atomic frequency and \( \lambda \) is the interaction constant. When we have the condition on the detuning, \( \delta = \omega_{eg} - \omega \),

\[
|\delta| > \sqrt{n + 1} \tag{2}
\]

for any “relevant” photon number, we can obtain an effective interaction Hamiltonian in the dispersive limit (see, for instance, [15])

\[
\hat{H}_{\text{eff}}^\chi = \chi \hat{a}^\dagger \hat{a} \hat{\sigma}_z, \quad \text{with} \quad \chi = \lambda^2 / \delta. \tag{3}
\]

Let us consider the atom initially in the following superposition

\[
|\psi_A(0)\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle) \tag{4}
\]

and the state of the field to be arbitrary, denoted by \(|\psi_F(0)\rangle\). If before the interaction we displace the unknown field by a quantity \( \alpha \), we have the state for the total initial wave function given by

\[
|\psi(\alpha; 0)\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle)|\psi_F(\alpha; 0)\rangle, \tag{5}
\]

where

\[
|\psi_F(\alpha; 0)\rangle = \hat{D}(\alpha)|\psi_F(0)\rangle
\]

with \( \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \)

is the displacement operator with amplitude \( \alpha \). By calculating the average value of the electric dipole \( \hat{\sigma}_x = \hat{\sigma}_- + \hat{\sigma}_+ \), we obtain [7]¹

\[
\langle \hat{\sigma}_x \rangle = \sum_{m=0}^{\infty} P_m(\alpha; 0) \cos(2\chi mt), \tag{6}
\]

where the photon distribution

\[
P_m(\alpha; 0) = \left| \langle \psi_F(\alpha; 0)|m\rangle \right|^2
\]

with \(|m\rangle \) a number state. By choosing \( t = \pi / (2\chi) \), we end up with

\[
\langle \hat{\sigma}_x \rangle = \sum_{m=0}^{\infty} P_m(\alpha; 0)(-1)^m \tag{7}
\]

that, except for a factor of \( \pi \) is the Wigner function [7,16].

The above treatment does not consider dissipation, and it is the aim of this Letter to study the case when we have a dissipative cavity. The problem of the interaction of a two-level atom with a quantized field in the dispersive regime in a cavity with losses was treated exactly by Peixoto and Nemes [15]. Here we will use a superoperator technique to solve it in an alternative way.

In the interaction picture, and in the dispersive approximation, the master equation that governs the dynamics of a two-level atom coupled with an electromagnetic field in a high-Q cavity is

\[
\frac{d}{dt} \hat{\rho} = -i [\hat{H}_{\text{eff}}^\chi, \hat{\rho}] + \hat{W} \hat{\rho}, \tag{8}
\]

where

\[
\hat{W} \hat{\rho} = 2\gamma \hat{a}^\dagger \hat{a} \hat{\rho} - \gamma \hat{a}^\dagger \hat{a} \hat{\rho} - \gamma \hat{a} \hat{a}^\dagger \hat{\rho}, \tag{9}
\]

and \( \hat{\rho} \) the density matrix of the system.

We define the superoperators

\[
\hat{L} \hat{\rho} = -\hat{F} \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{F}^\dagger \hat{a}\hat{a}^\dagger \hat{\rho}, \tag{10}
\]

\[
\hat{J} \hat{\rho} = 2\gamma \hat{a} \hat{a}^\dagger \hat{\rho}, \tag{11}
\]

¹ The average value of \( \hat{\sigma}_x \) can be written as \( \langle \hat{\sigma}_x \rangle = \text{Tr} [\hat{\sigma}_x \hat{\rho}] = \text{Tr} [\hat{R} \hat{\sigma}_x \hat{R}^\dagger \hat{\rho}] = \text{Tr} [\hat{R} \hat{\sigma}_x \hat{R}^\dagger \hat{\rho}] = \text{Tr} [\hat{\sigma}_x \hat{R}^\dagger \hat{R} \hat{\rho}], \) i.e., the expectation value of \( \hat{\sigma}_x \) (the atomic inversion or the probability of finding the atom in the excited state minus the probability of finding it in the ground state) for a rotated (in the atomic basis) density matrix (see, for instance, [7]). \( \hat{R} = \exp((\hat{\sigma}_- - \hat{\sigma}_+)\pi / 4) \).
where we have defined
\[ \hat{F} = \gamma \hat{1}_A + i \chi \hat{\sigma}_z, \]  
(12)
with \( \hat{1}_A = |e\rangle\langle e| + |g\rangle\langle g| \). It is not difficult to show that
\[ [\hat{J}, \hat{L}] = -\hat{S}_R \hat{J} \hat{\rho}, \]  
(13)
where the superoperator \( \hat{S}_R \) is defined as
\[ \hat{S}_R \hat{\rho} = \hat{F} \hat{\rho} + \hat{\rho} \hat{F}^\dagger. \]  
(14)
The solution to Eq. (8) subject to the initial state (5) is then given by
\[ \hat{\rho}(t) = e^{(\hat{L} + \hat{J})t} \hat{\rho}(\alpha; 0) = e^{\hat{L}t} e^{\hat{J} \hat{t}(\alpha)} \hat{\rho}(\alpha; 0), \]  
(15)
where
\[ \hat{f}(t) = \frac{1 - e^{-\hat{S}_R t}}{\hat{S}_R} \]  
(16)
and \( \hat{\rho}(\alpha; 0) = |\psi(\alpha; 0)\rangle \langle \psi(\alpha; 0)| \).

We need to operate the density matrix with the exponential of superoperators given above. It is not obvious how to operate on atomic states, and \( \hat{J} \) will operate only on atomic states, and \( \hat{f} \) will operate only in (5), and therefore we give an expression for it
\[ e^{\hat{J} \hat{t}(\alpha)} \hat{\rho}(0; \alpha) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{J}^n \left[ \hat{D}(\alpha)|\psi_F(0)\rangle \langle \psi_F(0)| \hat{D}^\dagger(\alpha) \right] \]
\[ \times \hat{f}^n \left[ |\psi_A\rangle \langle \psi_A| \right]. \]  
(17)
This is, because \( \hat{f} \) is an atomic superoperator, it will operate only on atomic states, and \( \hat{J} \) will operate only on field states. It is not difficult to show that
\[ 2 \hat{f}^n |\psi_A\rangle \langle \psi_A| = \frac{(1 - e^{-2 \eta \gamma \tau})^n}{(\xi + \xi^*)^n} \hat{1}_A \]
\[ + \frac{(1 - e^{-2 \eta \gamma \tau})^n}{(2 \xi)^n} |e\rangle\langle e| + \frac{(1 - e^{-2 \eta \gamma \tau})^n}{(2 \xi^*)^n} |g\rangle\langle g|, \]  
(18)
with \( \xi = \gamma + i \chi \). From (18), (17) and (15) we calculate \( \langle \dot{\sigma}_x \rangle \) and obtain
\[ \langle \dot{\sigma}_x \rangle = \frac{1}{2} \sum_{m=0}^{\infty} \left( \gamma \frac{1 - e^{-2 \eta \gamma \tau}}{\xi} \right)^m \]
\[ \times \sum_{k=0}^{\infty} e^{-2 \eta \gamma \tau (k+1)} (k+1) \hat{D}(\alpha)|\psi_F(0)\rangle |k+1\rangle |\hat{\rho}(\alpha; 0)\rangle |k+1\rangle + c.c. \]
(19)
By changing the summation index in the second sum of the above equation, with \( M = m + k \), we obtain
\[ \langle \dot{\sigma}_x \rangle = \frac{1}{2} \sum_{M=0}^{\infty} \left( \gamma \frac{1 - e^{-2 \eta \gamma \tau}}{\xi} \right)^M\]
\[ \times \sum_{M=m}^{\infty} e^{-2 M \xi \tau} \frac{(M)!}{(M-m)!} (M|\hat{\rho}(0; \alpha)\rangle |M\rangle + c.c. \]  
(20)
Finally, we can start the second sum of (20) from \( M = 0 \) (as we would only add zeros to the sum, because the factorial of a negative integer is infinite), and exchange the double sum in it, to sum first over \( m \), which gives
\[ \langle \dot{\sigma}_x \rangle = \frac{1}{2} \sum_{M=0}^{\infty} \left( \gamma \frac{1 - e^{-2 \eta \gamma \tau}}{\xi} \right)^M\]
\[ \times \sum_{M=0}^{\infty} e^{-2 M \xi \tau} \frac{(M)!}{(M-m)!} (M|\hat{\rho}(0; \alpha)\rangle |M\rangle + c.c. \]  
(21)
By defining
\[ \theta = \tan^{-1} \left( -\frac{\eta - e^{-2 \eta \gamma \tau} \sin(2 \tau) - \eta \cos(2 \tau)}{\eta^2 + e^{-2 \eta \gamma \tau} \cos(2 \tau) + \eta \sin(2 \tau)} \right) \]  
(22)
and
\[ \mu = \left( \frac{\eta^2 + e^{-4 \eta \gamma \tau} + 2 \eta e^{-2 \eta \gamma \tau} \sin(2 \tau)}{1 + \eta^2} \right)^{1/2}, \]  
(23)
with \( \tau = \chi t \) and \( \eta = \gamma / \chi \), we can have a final expression for \( \langle \dot{\sigma}_x \rangle \)
\[ \langle \dot{\sigma}_x \rangle = \sum_{M=0}^{\infty} \mu^M \cos(M \theta) \langle M|\hat{\rho}(0; \alpha)\rangle |M\rangle. \]  
(24)
Eq. (24) in general differs from an \( s \)-parametrized quasiprobability distribution. However, if we consider the case when \( \mu = 0 \), the only term that survives in (24) is \( M = 0 \), and we obtain the \( Q \)-function
\[ \langle \dot{\sigma}_x \rangle = (0|\hat{\rho}(0; \alpha)\rangle |0\rangle = |\alpha\rangle \hat{\rho}(0; \alpha) |\alpha\rangle = \pi Q(\alpha). \]  
(25)
To look for the value that makes \( \mu \) equal to zero, we rewrite, (23) for \( \tau = 3 \pi / 4 \) in the form
\[ \mu \left( \frac{3 \pi}{4} \right) = \frac{|\eta - e^{-3 \pi \gamma \tau / 2}|}{(1 + \eta^2)^{1/2}}, \]  
(26)
and therefore, for \( \eta = \exp(-3\eta\pi/2) \) we obtain \( \mu = 0 \). The value for \( \eta \) may be easily obtained numerically with the result \( \eta_{\mu=0} \approx 0.274457 \).

The parameter \( \gamma \) is the only one parameter that fixes all the other parameters: once known the rate at which the cavity decays, the cavity has to be tuned to obtain an effective interaction constant \( \chi \), given by \( \chi = \gamma/0.274457 \) and the atom has to traverse the cavity in a time \( t = 3\pi/(4\chi) \).

By direct measuring a quasiprobability distribution, namely, the Wigner function in the lossless case [7], i.e., for only one parameter (\( \gamma = 0 \)), or the \( Q \)-function in our case for a range of parameters (\( \gamma \approx 0.274457\chi \)), one can obtain complete information about the state of the field as quasiprobability distributions contain complete information of the density matrix [17]. Moreover, the Wigner and \( Q \) functions are related by integral or differential equations, for instance [16],

\[
W(\alpha) = e^{-\frac{1}{2} \chi} Q(\alpha). \tag{27}
\]

In conclusion, we have solved the dispersive interaction between a quantized electromagnetic field and a two level atom in the case of a real cavity (subject to losses) by utilizing superoperator techniques. We then have used those results to show that even in the dissipative case we can still obtain information about the initial cavity field by means of the \( Q \)-function (unlike the nondissipative case [7] where it is reconstructed the Wigner function). Both functions, being quasiprobability distributions contain complete information about the state of the cavity field. In this way, we have been able to extend the range of parameters in which complete information may be obtained in CQED, from \( \gamma = 0 \) to \( \gamma \approx 0.274457\chi \), i.e., by doing a right tuning complete information of the cavity field may be obtained despite of cavity losses.

References