Direct measurement of quasiprobability distributions in cavity QED

Raúl Juárez-Amaro and Héctor Moya-Cessa

1Instituto Nacional de Astrofísica, Óptica y Electrónica, Apartado Postal 51 y 216, 72000 Puebla, Puebla, Mexico
2Universidad Tecnológica de la Mixteca, Apartado Postal 71, 69000 Huajuapan de León, Oaxaca, Mexico

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We show that the set of $s$-parametrized quasiprobability distribution functions corresponding to an electromagnetic field in a cavity subject to dissipation can be directly measured. Such distributions contain whole information of the quantum state, therefore making it possible to recover information after losses have occurred.

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I. INTRODUCTION

The reconstruction of a quantum state is a central topic in quantum optics and related fields [1,2]. In the past, several techniques have been developed, for instance, the direct sampling of the density matrix of a signal mode in multiphoton optical homodyne tomography [3], tomographic reconstruction by unbalanced homodyning [4], reconstruction via photocounting [5], and cascaded homodyning [6], to cite a few. There have also been proposals to measure electromagnetic fields inside cavities [7,8]. The quantum state reconstruction in cavities is usually achieved through a finite set of selective measurements of atomic states [7], which makes it possible to construct quasiprobability distribution functions, such as the Wigner function, which constitute an alternative representation of a quantum state of the field. Nevertheless, in real experiments, the presence of noise and dissipation has normally destructive effects. Schemes that treat a lossy cavity have been proposed [8], which involve a physical process that allows the storage of information about the quantum coherences of the initial state in the diagonal elements of the density matrix of a transformed state. We will study the problem of reconstruction of the cavity field as studied in Ref. [7], but we will allow the cavity to have losses (i.e., a real cavity). We will then show that although it is not possible to reconstruct the Wigner function, it is still possible to recover whole information about the initial state. The importance of determining states of the quantized field are of great importance, in particular, given recent experimental achievements on the generation of Fock states in superconducting microwave cavities [9]. The method proposed here would represent one possible state-determination technique in the presence of dissipation. The relation between losses and quasiparametrized quasiprobability distributions has already been pointed out in Ref. [10] and problems with the reconstruction of the Wigner function have been analyzed in Ref. [11]. Methods to reconstruct the Wigner function in cavity QED are usually based on the expression

$$W(\alpha) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle \alpha, n | \hat{\rho} | \alpha, n \rangle,$$

where $\hat{\rho}$ is the density matrix of the system and $|\alpha, n\rangle$ are the so-called displaced Fock or number states [12]. The expression above allows the reconstruction of the Wigner function by simply displacing the density matrix and measuring its diagonal elements. The difficulty to reconstruct the Wigner function in the dissipative regime arises from the fact that the sum in the equation above has no weight that allows to cut it by neglecting higher-order terms. Here, we will show that dissipation introduces a weight in the sum not allowing anymore the reconstruction of the Wigner function but of some other quasiprobability distributions. In Sec. II, we analyze the interaction between a two-level atom and a quantized field in the dispersive regime in the presence of dissipation. In Sec. III, we show that when losses are taken into account, it is still possible to recover the whole information about the initial field. Section IV is devoted to conclusions.

II. DISPERSIVE REGIME IN THE PRESENCE OF DISSIPATION

The Hamiltonian for the interaction between a quantized field and a two-level atom reads (we have set $\hbar = 1$)

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \frac{\omega_{eg}}{2} \hat{\sigma}_z + \lambda (\hat{a}^\dagger \hat{a} \hat{\sigma}_+ + \hat{a} \hat{a}^\dagger \hat{\sigma}_-),$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the field mode, respectively, obeying $[\hat{a}, \hat{a}^\dagger] = 1$. $\hat{\sigma}_+ = |e\rangle \langle g|$ and $\hat{\sigma}_- = |g\rangle \langle e|$ are the raising and lowering atomic operators, respectively, $|e\rangle$ being the excited state and $|g\rangle$ the ground state of the two-level atom. The atomic operators obey the commutation relation $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$. $\omega$ is the field frequency, $\omega_{eg}$ the atomic frequency, and $\lambda$ is the interaction constant. When we have the condition on the detuning, $\delta = \omega_{eg} - \omega$, $|\delta|/\hbar > \sqrt{n + 1}$ for any “relevant” photon number, we can obtain an effective interaction Hamiltonian in the dispersive limit (see, for instance, Ref. [13]),

$$\hat{H}_{\text{eff}}^{\text{disp}} = \chi \hat{a}^\dagger \hat{a} \hat{\sigma}_z,$$

with $\chi = \lambda^2 / \delta$. In the interaction picture and in the dispersive approximation, the master equation that governs the dynamics of a two-level atom coupled with an electromagnetic field in a high-$Q$ cavity [13] is
The solution to Eq. (4), subject to the initial state \( \hat{\rho}(0) \), is then given by

\[
\hat{\rho}(t) = e^{(\hat{L} + \hat{J}^\dagger)} \hat{\rho}(0) = e^{\hat{L}t} e^{\hat{J}^\dagger} \hat{\rho}(0),
\]

where

\[
\hat{J}(t) \hat{\rho} = \frac{1 - e^{-\hat{S}_F t}}{\hat{S}_F} \hat{\rho}.
\]

and \( \hat{\rho}(0) = |\psi(0)\rangle \langle \psi(0)| \). Let us consider the atom initially in the following superposition:

\[
|\psi_A(0)\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle),
\]

and the state of the field to be arbitrary, denoted by \( |\psi_F(0)\rangle \).

To obtain the evolved density matrix, we need to operate the density matrix with the exponential of superoperators given above. It is not obvious how \( e^{\hat{J}(t)} \) will apply on the total initial state, therefore we give an expression for it,

\[
e^{\hat{J}(t)} \hat{\rho} = \sum_{n=0}^{\infty} \frac{\hat{\rho}_F(0)^n}{n!} \hat{J}^n |\psi_A(0)\rangle \langle \psi_A|,
\]

with \( \hat{\rho}_F(0) = |\psi_F(0)\rangle \langle \psi_F(0)| \). Because \( \hat{J} \) is an atomic superoperator, it will operate only on atomic states, and \( \hat{J}^\dagger \), being a field superoperator, will operate only on field states. It is not difficult to show that

\[
\hat{J}^n |\psi_A\rangle \langle \psi_A| = \frac{1}{2} \left[ (1 - e^{-\xi t})^n |e\rangle + (1 - e^{-2\xi t})^n |e\rangle \right.
\]

\[
\times \langle g | + \frac{1}{2} \left( 1 - e^{-2\xi t} \right)^n |g\rangle \langle e | \right]
\]

and

\[
\hat{J}^n \hat{\rho}_F(0) = (2 \gamma)^n \hat{\rho}_F(0) (\hat{a}^\dagger)^n,
\]

with \( \xi = \gamma + i \chi \). Therefore,

\[
e^{\hat{J}(t)} \hat{\rho} = e^{-\Gamma a a^\dagger} \hat{\rho} e^{-\Gamma a^\dagger a},
\]

By using

\[
we may finally calculate \( \langle \hat{S}_A \rangle \) and obtain

\[
\langle \hat{S}_A \rangle = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(\gamma(1 - e^{-\xi t}/\xi)^m) m!}{m!} \sum_{k=0}^{\infty} e^{-2\xi k} (m+k)! k! \times \langle k+m | \hat{\rho}_F(0) | k+m \rangle + c.c.
\]

By changing the summation index in the second sum of the above equation with \( n = m + k \), we obtain

\[
\langle \hat{S}_A \rangle = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\gamma e^{2\xi t} - 1)/\xi^n}{m!} \times \sum_{n=m}^{\infty} e^{-2n\xi t} \frac{(n)!}{(n-m)!} \langle n | \hat{\rho}_F(0) | n \rangle + c.c.
\]

Finally, we can start the second sum of Eq. (20) from \( n = 0 \) (as we would only add zeros to the sum because the factorial of a negative integer is infinite) and exchange the double sum in it to sum first over \( m \), which gives

\[
\langle \hat{S}_A \rangle = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\gamma e^{2\xi t} - 1)/\xi^n}{m!} \times \sum_{n=m}^{\infty} e^{-2n\xi t} \frac{(n)!}{(n-m)!} \langle n | \hat{\rho}_F(0) | n \rangle + c.c.
\]

By defining

\[
\tan \theta = -\frac{\eta + e^{-2\eta} \gamma [\sin(2\tau) - \eta \cos(2\tau)]}{\eta^2 + e^{-2\eta} \gamma [\cos(2\tau) + \eta \sin(2\tau)]},
\]

and

\[
\mu = \left( \eta^2 + e^{-4\eta} + 2\eta e^{-2\eta} \sin(2\tau) \right)^{1/2},
\]
with \( \tau = \chi t \) and \( \eta = \gamma / \chi \), we can have a final expression for \( \langle \hat{\sigma}_x \rangle \),

\[
\langle \hat{\sigma}_x \rangle = \sum_{n=0}^{\infty} \mu^n \cos(n \theta) \langle n | \hat{\rho}_F(0) | n \rangle.
\]  \hspace{1cm} (24)

In Fig. 1, we plot \( \langle \hat{\sigma}_x \rangle \) as a function of \( \tau \) for an initial coherent state of the field,

\[
|\alpha\rangle = \hat{D}(\alpha) |0\rangle = e^{-|\alpha|^2 / 2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
\]  \hspace{1cm} (25)

The effects of dissipation are clearly seen in the reduction of the amplitude of the revival.

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![Graph](image1.png)

**FIG. 1.** We plot \( \langle \hat{\sigma}_x \rangle \) as a function of \( \tau \) for a coherent state with \( \alpha = 4 \) and for \( \eta = 0 \) (solid line) and \( \eta = 0.1 \) (dashed line).

![Graph](image2.png)

**FIG. 2.** We plot \( \mu \) as a function of \( \tau \) for \( \eta = 0 \) (solid line) and \( \eta = 0.1 \) (dashed line). It may be seen that \( \mu = 0.6931 \), i.e., \( s \approx -0.1812 \) for \( \tau \approx 1.689 \).

![Graph](image3.png)

**FIG. 3.** (a) Plot of the Wigner function, i.e., \( s = 0 \) quasiprobability distribution for the state \(|\psi_F(0)\rangle = b_0 |0\rangle + \sqrt{1-b_0^2} |1\rangle \) and \( b_0 = 0.2 \), and (b) plot of the \( s \approx -0.1812 \) quasiprobability distribution function. \( X = \text{Re}[\alpha] \) and \( Y = \text{Im}[\alpha] \).

### III. Quasiprobabilities and Losses

If instead of considering a coherent state in Eq. (24), we consider an initial arbitrary state displaced by an amount \( \alpha \), i.e., \( \hat{D}(\alpha) |\psi_F(0)\rangle \), we obtain

\[
\langle \hat{\sigma}_x \rangle = \sum_{n=0}^{\infty} \mu^n \cos(n \theta) \langle n | \hat{D}(\alpha) \hat{\rho}_F(0) \hat{D}(\alpha) | n \rangle.
\]  \hspace{1cm} (26)

By choosing an interaction time such that \( \theta = -\pi \), Eq. (26) reduces to

\[
\langle \hat{\sigma}_x \rangle = \sum_{n=0}^{\infty} (-\mu)^n \langle n | \hat{D}(\alpha) \hat{\rho}_F(0) \hat{D}(\alpha) | n \rangle.
\]  \hspace{1cm} (27)
One may numerically obtain the value of \( \tau \approx 1.689 \) for \( \theta = -\pi \) in the case \( \eta = 0.1 \), which we will use in the following figures. In Fig. 2, we plot \( \mu \) as a function of \( \tau \). As soon as \( \mu \neq 1 \), \( \mu^n \) becomes smaller than unity producing errors if one wants to reconstruct the Wigner function. However, one can determine completely the state by noting that an \( s \)-parametrized quasiprobability may be reconstructed exactly. It should be stressed that once determined in an experiment, the values of the interaction constant and of the decay rate set the interaction time with the condition \( \theta = -\pi \), which finally sets the value of \( \mu \) and therefore the quasiprobability to measure. By setting \( \mu = (1 + s)/(1 - s) \) (the \( \mu \) that corresponds to the value of \( \tau = 1.689 \), i.e., \( \eta = 0.1 \)), we may finally cast Eq. (27) as an \( s \)-parametrized quasiprobability distribution function \( F(\alpha, s) \) [14],

\[
\langle \hat{\sigma}_r \rangle = \sum_{n=0}^{\infty} \left( \frac{s + 1}{s - 1} \right)^n \langle \alpha, n | \hat{\rho}_F(0) | \alpha, n \rangle = \frac{\pi(1-s)}{2} F(\alpha, s),
\]

which shows a relation between quasiprobability distributions and a simple measurement of the atomic polarization operator \( \hat{\sigma}_r \) in the case of cavity losses.

In Fig. 3(a), we plot the Wigner function for the state \( |\psi_F(0)\rangle = b_0 |0\rangle + \sqrt{1 - b_0^2} |1\rangle \), with \( b_0 = 0.2 \), and in Fig. 3(b), \( s = -0.1812 \) parametrized quasiprobability distribution that corresponds to the parameters \( \tau = 1.689 \) and \( \eta = 0.1 \) for the same state. One may see that the reconstructed quasiprobability distribution is not as negative as the Wigner function. Of course, for greater values of the decay parameter, the effect would be stronger. However, even in the case of dissipation one would measure a negative quasiprobability distribution, but it should be stressed that one would not measure the Wigner function.

IV. CONCLUSIONS

We have solved the dispersive interaction between a quantized electromagnetic field and a two-level atom in the case of a real cavity (subject to losses) by using superoperator techniques. We have then used those results to show that even in the dissipative case we can obtain information about the initial cavity field by means of \( s \)-parametrized quasiprobability distribution function. Such functions contain complete information about the state of the cavity field. One thing to bear in mind is the fact that the effective interaction constant is much smaller than the real interaction constant, \( \chi \ll \lambda \). Finally, note that even for a superposition of \( |0\rangle \) and \( |1\rangle \), both quasiprobabilities already show a difference. It is expected that for more intense fields, the difference between the ideal quasiprobability to measure (the Wigner function) and the quasiprobability measured will be of great importance.

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