Introduction to quantum optics: quasiprobability distribution functions

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1. Introduction

In this Chapter we do a brief introduction to quantum optics. It is thought to be self contained and has as main topic quasiprobability distribution functions as well as some cavity quantum electrodynamic (CQED) systems that are difficult to find in text books. It may be used for a one-semester course of (Special Topics on) Quantum Optics. In the first section we do a survey of some of the simple algebra we use throughout the chapter. In Section II we introduce the harmonic oscillator and some of its states and there we also see the need of treating different orders of ladder operators, namely normal and antinormal orders. In Section III, we introduce from a correspondence principle approach the main quasiprobability distribution functions: the Wigner function, the Husimi \textit{Q}-function and the Glauber-Sudarshan \textit{P}-function. In Section IV more states of the harmonic oscillator are introduced. Sections V and VI treat CQED like systems, namely the ion-laser and mirror-filed interactions, respectively. Finally in Section VII we do a brief review of the phase problem.

1.1. Schrödinger Equation

The main task in quantum mechanics is to solve the Schrödinger equation

$$\frac{d|\psi\rangle}{dt} = \frac{1}{i\hbar} \hat{H} |\psi\rangle,$$

(1)

In order to achieve this, sometimes it is convenient to perform unitary transformations such that we may simplify the problem. For instance we may do $|\psi\rangle = \hat{T}|\phi\rangle$ and obtain a new equation for $|\phi\rangle$ (with $\hat{T} = e^{-i\xi \hat{A}}$, and $\hat{A}$ a Hermitian operator)

$$\frac{d|\phi\rangle}{dt} = \frac{1}{i\hbar} \hat{H}_T |\phi\rangle$$

(2)

where

$$\hat{H}_T = \hat{T}^\dagger \hat{H} \hat{T} = e^{i\xi \hat{A}} \hat{H} e^{-i\xi \hat{A}}.$$  

(3)

Developing the exponentials in Taylor series and grouping terms we obtain

$$e^{i\xi \hat{A}} \hat{H} e^{-i\xi \hat{A}} = \hat{H} + i\xi [\hat{A}, \hat{H}] + \frac{(i\xi)^2}{2!} [\hat{A}, [\hat{A}, \hat{H}]] + \frac{(i\xi)^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{H}]]] + ....$$  

(4)

This equation is valid for any two operators $\hat{H}$ and $\hat{A}$.

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If we do $\hat{A} \to \hat{p}$ and $\hat{H} \to \hat{q}$ equation (4) shows that $e^{i\alpha\hat{p}}$ displaces the position operator
\[
f(\hat{q} + \alpha) = e^{i\alpha/\hbar} f(\hat{q}) e^{-i\alpha/\hbar}.
\]
where we have used the commutator $[\hat{q}, \hat{p}] = i\hbar$. A more general operator $e^{-i(\hat{q}\hat{p} - \hat{p}\hat{q})/\hbar}$ produces displacements in both $\hat{q}$ and $\hat{p}$ simultaneously
\[
f(\hat{q}, \hat{p}) = e^{i(\hat{q}\hat{p} - \hat{p}\hat{q})/\hbar} f(\hat{q} + q_0, \hat{p} + p_0) e^{-i(\hat{q}\hat{p} - \hat{p}\hat{q})/\hbar}.
\]

1.2. Von Neumann equation

An equation that is equivalent to the Schrödinger equation and is commonly used because its form allows to take into account easily the effects of the environment is the so called Von Neumann equation. It is obtained from Eq. (1) by multiplying it by the 'bra' $\langle \psi |$ by the right
\[
\frac{d}{dt}\langle \psi | = \frac{1}{i\hbar} \hat{H} |\psi \rangle,
\]
and adding it with the adjoint of Eq. (1) multiplied by the 'ket' $|\psi \rangle$ by the left, so we obtain
\[
\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}],
\]
with $\hat{\rho} = |\psi \rangle \langle \psi |$ the system’s density matrix. $[\hat{H}, \hat{\rho}] = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}$ is the commutator between $\hat{H}$ and $\hat{\rho}$.

1.3. Baker-Hausdorff Formula

Equation 6) has complicated terms in the sense that it has exponentials of the sum of non-commuting operators. They may be easily factorized in the product of (three) exponentials because of the "simplicity" of the commutation relation of the operator involved, namely, position and momentum. In order to realize such factorization we use what is known in the literature as the Baker-Hausdorff formula, that refers that, if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ then
\[
e^{\hat{A} + \hat{B}} = e^{-\frac{[\hat{A}, \hat{B}]}{2}} e^{\hat{A}} e^{\hat{B}}.
\]
so that
\[
e^{i(\hat{q}\hat{p} - \hat{p}\hat{q})/\hbar} = e^{i\frac{q_0 p_0}{\hbar}} e^{-i\frac{q_0 p_0}{\hbar}} e^{i\frac{q_0 p_0}{\hbar}}.
\]

In order to prove Baker-Hausdorff’s formula we write
\[
\hat{F}(\lambda) = e^{\lambda(\hat{A} + \hat{B})} = e^{f(\lambda)\hat{A}} e^{g(\lambda)\hat{B}},
\]
and derive $\hat{F}(\lambda)$ with respect to $\lambda$
\[
\frac{d\hat{F}(\lambda)}{d\lambda} = (\hat{A} + \hat{B})\hat{F}(\lambda).
\]
From the first equality
\[
\frac{d\hat{F}(\lambda)}{d\lambda} = \left( f'(\lambda) + g'(\lambda)\hat{A} + h'(\lambda) e^{g(\lambda)\hat{A}} \hat{B} e^{-g(\lambda)\hat{A}} \right) \hat{F}(\lambda),
\]
from the second. We have introduced in the former equation a unity operator given by $e^{-g(\lambda)\hat{A}} e^{g(\lambda)\hat{A}}$. We use now (4) to calculate
\[
e^{g(\lambda)\hat{A}} \hat{B} e^{-g(\lambda)\hat{A}} = \hat{B} + g[\hat{A}, \hat{B}]
\]
so Eq. (13) may be rewritten as
\[
\frac{d\hat{F}(\lambda)}{d\lambda} = \left( f'(\lambda) + g'(\lambda)\hat{A} + h'(\lambda) \left\{ \hat{B} + g[\hat{A}, \hat{B}] \right\} \right) \hat{F}(\lambda).
\]
Equating with (12) to obtain the system of differential equations
\[ g'(\lambda) = 1, \quad h'(\lambda) = 1, \quad h'(\lambda)g(\lambda)[\hat{A}, \hat{B}] + f'(\lambda) = 0 \] (16)
that has as solution
\[ g(\lambda) = \lambda + g(0), \quad h(\lambda) = \lambda + h(0), \quad f(\lambda) = -\left(\frac{\lambda^2}{2} + g(0)\lambda\right)[\hat{A}, \hat{B}] + f(0). \] (17)
By evaluating (11) in zero, we obtain the initial conditions
\[ f(0) = 0, \quad g(0) = 0 \text{ and } h(0) = 0, \] to finally obtain
\[ \text{Baker-Hausdorff's theorem} \]
\[ \hat{F}(\lambda = 1) = e^{\hat{A} + \hat{B}} = e^{-\frac{[\hat{A}, \hat{B}]}{2}}e^{\hat{A}}e^{\hat{B}}. \] (18)

2. Quantum mechanical harmonic oscillator

The Hamiltonian for the harmonic oscillator is written as (we consider unity mass for simplicity)
\[ \hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2), \] (19)
we define the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ known also as the excitation annihilation and creation operators
\[ \hat{a} = \sqrt{\frac{\omega}{2\hbar}} \hat{q} + \frac{1}{\sqrt{2\hbar\omega}} \hat{p}, \quad \hat{a}^\dagger = \sqrt{\frac{\omega}{2\hbar}} \hat{q} - \frac{1}{\sqrt{2\hbar\omega}} \hat{p}. \] (20)
This operators obey the commutation relation
\[ [\hat{a}, \hat{a}^\dagger] = 1, \] (21)
such that the Hamiltonian (19) may be written in terms of the ladder operators as
\[ \hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \] (22)
by defining the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ we finally write
\[ \hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}). \] (23)

2.1. Fock states

The eigenstates of (23) are known as Fock or number states $|n\rangle$ with eigenvalues $\hbar\omega(n + \frac{1}{2})$, i.e.
\[ \hat{H} |n\rangle = \hbar\omega(\hat{n} + \frac{1}{2}) |n\rangle \] (24)
where $n$ is an integer non-negative number, and it is identified with the number of excitations (photons in the case of electromagnetic field)

Fock states are also eigenstates of the number operator
\[ \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle, \] (25)
the vacuum state of the harmonic oscillator is defined from
\[ \hat{a} |0\rangle = 0. \] (26)
In fact the creation and annihilation operators act on the number state in the following form
\[ \hat{a}^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n - 1\rangle. \] (27)
Any state vector $|n\rangle$ may be obtained from the vacuum $|0\rangle$ via the creation operator,

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle.$$  \hfill (28)

Number states form an orthonormal basis

$$\langle n|m \rangle = \delta_{nm}. \hfill (29)$$

The unity operator may be written in terms of number states as

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1.$$ \hfill (30)

By using the completeness relation it is possible to express any operator in terms of number states, for instance the annihilation operator

$$\hat{a} = \sum_{n=0}^{\infty} \hat{a} |n\rangle \langle n| = \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle \langle n+1|,$$ \hfill (31)

and the creation operator

$$\hat{a}^\dagger = \sum_{n=0}^{\infty} \hat{a}^\dagger |n\rangle \langle n| = \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle \langle n|.$$ \hfill (32)

The number operator has the diagonal decomposition

$$\hat{n} = \sum_{n=0}^{\infty} n |n\rangle \langle n|.$$ \hfill (33)

Number states have no uncertainty in intensity

$$\langle \Delta n \rangle \equiv \sqrt{\langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2} = 0.$$ \hfill (34)

Averages for position and momentum are null, $\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$, however their uncertainties are not

$$\langle \Delta \hat{q} \rangle = \sqrt{\langle n|\hat{q}^2|n\rangle} = \sqrt{\frac{(2n+1)\hbar}{2\omega}}, \quad \langle \Delta \hat{p} \rangle = \sqrt{\langle n|\hat{p}^2|n\rangle} = \sqrt{\frac{(2n+1)\hbar^2}{2}},$$ \hfill (35)

such that they are minimized only for the vacuum. The equation

$$P(n) = \langle \langle n|\psi\rangle|^2,$$ \hfill (36)

gives the probability to have $n$ number of excitations (photons) in the state $|\psi\rangle$.

### 2.2. Coherent states

We may build arbitrary superpositions of number states to obtain new states, in particular we can construct coherent states of the harmonic oscillator. They may be obtained in different forms: as eigenstates of the annihilation operator, or equivalently for the harmonic oscillator, as states whose averages follow the classical trajectories of position, momentum and energy \[1\] or as displacement of the vacuum.

Let us define them as eigenstates of the annihilation operator

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle \hfill (37)$$

because $\hat{a}$ is a non-Hermitian operator its eigenvalues, \(\alpha\) are complex. Other properties of these states may be obtained by using the Glauber displacement operator \[2\] \[\hat{D}(\alpha) = e^{-i(\hat{p}\hat{q} - \hat{q}\hat{p})/\hbar}, \text{[see Eq. (6)]}\]

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}, \hfill (38)$$

with $\alpha = \sqrt{\frac{\hbar}{2\omega}} q_0 + i \frac{\hbar p_0}{\sqrt{2\omega}}$.

The displacement operator has the following properties (6)

$$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha), \quad \hat{D}(\alpha + \beta) = \hat{D}(\alpha) \hat{D}(\beta) e^{-i\hbar m(\alpha \beta^*)} \hfill (39)$$

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and
\[ \hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \quad \hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*, \]
(40)
to name some. Coherent states, \(|\alpha\rangle\), may be generated by application of the displacement operator on the vacuum
\[ |\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle, \]
(41)
that, by developing the exponential in Taylor series gives
\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]
(42)
is obtained. In this equation the value \(|\alpha|^2\) represents the average value of excitation \(\hat{n}\) in the coherent state \(|\alpha\rangle\):
\[ \hat{n} = \langle\hat{n}|\hat{a}\hat{a}|\alpha\rangle = \langle\hat{a}\hat{a}|\alpha\rangle = |\alpha|^2. \]
(43)

The solution for the harmonic oscillator Hamiltonian for an initial coherent state is given in a very simple form:
\[ |\psi(t)\rangle = e^{-i\hat{H}t}\hat{a}|\alpha\rangle = e^{-i\hat{H}t}e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \]
(44)
introducing \(e^{-i\hat{H}t}\) into the sum and using the fact that the states \(|k\rangle\) are eigenstates of the number operators \(\hat{n}\) we have
\[ |\psi(t)\rangle = e^{-i\hat{H}t}e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |k\rangle \equiv e^{-i\hat{H}t}|\alpha e^{-i\hat{H}t}\rangle, \]
(45)
this is, a coherent state that rotates with the harmonic oscillator frequency.

Coherent states are eigenstates of the annihilation operator, but, How does act the creation operator on them? This question may be answered by using the coherent density matrix \(|\alpha\rangle\langle\alpha|\)
\[ \hat{a}^\dagger|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\alpha^n(\alpha^*)^k}{\sqrt{n!k!}} \sqrt{n+1}|n+1\rangle\langle k| \]
(46)
we note that
\[ \left( \frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle\langle\alpha| = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\alpha^n(\alpha^*)^k}{\sqrt{n!k!}} \sqrt{n+1}|n+1\rangle\langle k| \]
(47)
so that
\[ \hat{a}^\dagger|\alpha\rangle\langle\alpha| = \left( \frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle|\alpha\rangle. \]
(48)

Analogously
\[ |\alpha\rangle\langle\alpha|\hat{a} = \left( \frac{\partial}{\partial \alpha^*} + \alpha \right) |\alpha\rangle\langle\alpha|. \]
(49)

This expression is of interest, particularly when we pass from Master equations to Fokker-Planck equations [3]. We will use this expression Section III where we relate quasiprobabilities in a differential form.

Coherent states form an over-complete set of states. The identity operator is written in terms of coherent states as
\[ \frac{1}{\pi} \int |\alpha\rangle \langle\alpha| d^2\alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n\rangle\langle m| \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} \frac{r^{n+m}}{n!m!} e^{i(n-m)\theta} r dr d\theta = \sum_n |n\rangle\langle n| \]
(50)
where we have set $\alpha = re^{i\theta}$. By using the above completeness relation, the annihilation operator may be expressed as
\[
\hat{a} = \frac{1}{\pi} \int \hat{a}|\alpha\rangle \langle \alpha| d^2\alpha = \frac{1}{\pi} \int \alpha |\alpha\rangle \langle \alpha| d^2\alpha
\]
and the creation operator
\[
\hat{a}^\dagger = \frac{1}{\pi} \int \alpha^* |\alpha\rangle \langle \alpha| d^2\alpha.
\]

The number operator is a bit more complicated to express in terms of coherent states because
\[
\hat{n} = \hat{a}^\dagger \hat{a} = \frac{1}{\pi^2} \int |\beta\rangle \langle \beta| \hat{a}^\dagger \hat{a} (1) |\alpha\rangle \langle \alpha| d^2\alpha = \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2\alpha = 1
\]
(51)
\[
\text{this is as a representation that includes off-diagonal terms. However, note that if we write the operator } \hat{a}^\dagger \hat{a} = \hat{a} \hat{a}^\dagger - 1, \text{ we can do}
\]
\[
\hat{n} = \hat{a}^\dagger \hat{a} = \frac{1}{\pi} \int \hat{a} |\alpha\rangle \langle \alpha| d^2\alpha - 1 = \frac{1}{\pi} \int |\alpha|^2 |\alpha\rangle \langle \alpha| d^2\alpha - 1
\]
(52)
i.e. a diagonal form. This implies that the 
ordering of the annihilation and creation operators is of importance. We will look in more detail the ordering of such operators in the Subsection E and their importance in the Section on quasiprobabilities.

The excitation number for the coherent states is given by the Poissonian
\[
P(n) = |\langle n |\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!},
\]
(53)
In Fig. 1 we plot such a distribution.

The excitation uncertainty for coherent states is given by
\[
\langle \Delta \hat{n} \rangle = \sqrt{\langle \hat{n}^2 |\alpha\rangle - \langle \hat{n} |\alpha\rangle^2} = \sqrt{\langle \hat{a}^\dagger \hat{a} |\alpha\rangle \langle \hat{a}^\dagger \hat{a} |\alpha\rangle} = |\alpha|^2.
\]
(54)
Finally, we note that the average of operators for position, momentum and energy for coherent states follow classical physics [1]
\[
\langle \alpha |\hat{q}|\alpha\rangle = q_c, \quad \langle \alpha |\hat{p}|\alpha\rangle = p_c, \quad \langle \alpha |\hat{H}|\alpha\rangle = H_c
\]
(55)
where the index $c$ means classical variables. Because of this, coherent states are called quasi-classical states, and are a reference to other states of the harmonic oscillator. The uncertainties for position and momentum may be found to be
\[
\langle \Delta \hat{q} \rangle = \sqrt{\frac{\hbar}{2\omega}}, \quad \langle \Delta \hat{p} \rangle = \sqrt{\frac{\hbar\omega}{2}}
\]
(56)
such that
\[
\langle \Delta \hat{q} \rangle \langle \Delta \hat{p} \rangle = \frac{\hbar}{2}
\]
(57)
i.e. coherent state minimize the uncertainty principle.

2.3. Displaced number states

There exist many more states of the harmonic oscillator, among them the so-called displaced number states which may be obtained by the application of the displacement operator onto the number state
\[
|\alpha, n \rangle = \hat{D}(\alpha)|n\rangle.
\]
(58)
As they are orthonormal, $\langle \alpha, m |\alpha, n \rangle = \delta_{m,n}$ they form a complete basis
\[
\sum_{n=0}^{\infty} |\alpha, n \rangle \langle \alpha, n| = 1.
\]
(59)
They excitation number distribution is given by

\[ P_m = e^{-|\alpha|^2} \frac{m!}{m!} |\alpha|^{2(m-n)}(L_m^{m-n}(|\alpha|^2))^2, \quad m \geq n \]  \hspace{1cm} (62)

and

\[ P_m = e^{-|\alpha|^2} \frac{m!}{m!} |\alpha|^{2(n-m)}(L_m^{n-m}(|\alpha|^2))^2, \quad n \geq m \]  \hspace{1cm} (63)

In Fig. 2 we plot several distributions of these states for different values of \( \alpha \) and \( n \).

These states will be of importance in next Section where we talk about quasiprobability distribution functions.

2.4. Phase states

There exist several kinds of states that are not normalized, such as position eigenstates, momentum eigenstates and also phase states. The latter ones may be defined as

\[ |\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle. \]  \hspace{1cm} (64)

These states are eigenstates of the so-called Susskind-Glogower (phase) operator [4,5]

\[ \hat{V} = \frac{1}{\sqrt{n+1}} \hat{a} \]  \hspace{1cm} (65)

i.e.

\[ \hat{V} |\phi\rangle = e^{i\phi} |\phi\rangle. \]  \hspace{1cm} (66)

We can write the unity operator also in terms of phase states

\[ \int_{-\pi}^{\pi} |\phi\rangle \langle \phi| d\phi = 1, \]  \hspace{1cm} (67)

therefore we can use them to perform traces. A property of \( \hat{V} \) is that \( \hat{V} \hat{V}^\dagger = 1 \) but \( \hat{V}^\dagger \hat{V} \neq 1 \). This may be better seen using the expansion of the operator in the Fock basis

\[ \hat{V} = \sum_{n=0}^{\infty} |n\rangle \langle n+1|. \]  \hspace{1cm} (68)

In Section VII we study with some more detail the quantum phase.

2.5. Ordering of ladder operators

In some problems in quantum mechanics it is needed to calculate functions of the operator \( \hat{n} = \hat{a}^\dagger \hat{a} \) where \( \hat{a} \) and \( \hat{a}^\dagger \) are annihilation and creation operators of the harmonic oscillator, respectively.

Here we obtain an expression for \( \hat{n}^k \) in normal and anti-normal order, i.e. a sum of coefficients multiplying normal and anti-normal ordered forms of \( \hat{a} \) and \( \hat{a}^\dagger \). This allows us to obtain an expression for functions of the operator \( \hat{n} \) and demonstrate as a particular example a lemma given by Louissel [6] for the exponential of the number operator.
2.5.1. Normal ordering

One may use the commutation relations of the annihilation and creation operators to obtain the powers of $\hat{n}$ in normal, anti-normal or symmetric order. For instance, we can express $\hat{n}^2$ in normal order, for $k = 2$ as

$$\hat{n}^2 = [\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a},$$

for $k = 3$ as

$$\hat{n}^3 = [\hat{a}^\dagger]^3 \hat{a}^3 + 3[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a},$$

and for $k = 4$

$$\hat{n}^4 = [\hat{a}^\dagger]^4 \hat{a}^4 + 6[\hat{a}^\dagger]^3 \hat{a}^3 + 7[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a},$$

where the coefficients multiplying the different powers of the normal ordered operators do not show an obvious form to be determined. In writing the above equations we have used repeatedly the commutator $[\hat{a}, \hat{a}^\dagger] = 1$.

From the above it is not difficult to show that

$$\hat{n}^k = \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m,$$  

(72)

with [8]

$$S_k^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \frac{m!}{j!(m-j)!} j^k,$$  

(73)

the Stirling numbers of the second kind. We now write a function of $\hat{n}$ in a Taylor series as

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \hat{n}^k,$$  

(74)

and inserting (72) in this equation we obtain

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m.$$  

(75)

Because $S_k^{(m)} = 0$ for $m > k$ we can take the second sum in (75) to infinite and interchange the sums to have

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}.$$  

(76)

For the same reason stated above, we may start the second sum at $k = m$,

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=m}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}.$$  

(77)

By noting that

$$\frac{\Delta^m f(x)}{m!} = \sum_{k=m}^{\infty} \frac{f^{(k)}(x)}{k!} S_k^{(m)},$$  

(78)

where $\Delta$ is the difference operator, defined as [8]

$$\Delta^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} f(x + k),$$  

(79)

we may write (77) as

$$f(\hat{n}) = \sum_{m=0}^{\infty} \frac{[\hat{a}^\dagger]^m \hat{a}^m \Delta^m}{m!} f(0) \equiv e^{\hat{n}} : f(0)$$  

(80)

where $: \hat{n} :$ stands for normal order.
2.5.2. Lemma 1

If we choose the function \( f(\hat{n}) = \exp(-\gamma \hat{n}) \), we have that

\[
\Delta^m f(0) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} e^{-\gamma k},
\]

and then we obtain the well-known lemma [6]

\[
e^{-\gamma \hat{n}} = e^{(e^{-\gamma} - 1)\hat{n}}.
\]  

2.5.3. Antinormal ordering

Following the procedure introduced in the former section, we can write \( \hat{n}^k \) in antinormal order as

\[
\hat{n}^k = (-1)^k \sum_{m=0}^{k} (-1)^m S^{(m+1)}_{k+1} \hat{a}^m \hat{a}^\dagger^m,
\]

and a function of the number operator as

\[
f(\hat{n}) = \sum_{m=0}^{\infty} (-1)^m \hat{a}^m \hat{a}^\dagger^m \sum_{k=m}^{\infty} (-1)^k \frac{f^{(k)}(0)}{k!} S^{(m+1)}_{k+1}.
\]

The second sum differs from (78) in the extra \((-1)^k\) and the parameters of the Stirling numbers. We can define \( u = -x \), such that \( f^{(k)}(x)_{x=0} = (-1)^k f^{(k)}(u)_{u=0} \), and use the identity [8]

\[
S^{(m+1)}_{k+1} = (m+1)S^{(m+1)}_{k+1} + S^{(m)}_k
\]

to write

\[
f(\hat{n}) = \sum_{m=0}^{\infty} (-1)^m \hat{a}^m \hat{a}^\dagger^m \left( (m+1) \sum_{k=m}^{\infty} \frac{f^{(k)}(u=0)}{k!} S^{(m+1)}_{k+1} + \sum_{k=m}^{\infty} \frac{f^{(k)}(u=0)}{k!} S^{(m)}_k \right)
\]

so we can use again Eq. (78) to finally write

\[
f(\hat{n}) = (1 + \Delta) e^{-\Delta \hat{n}} f(0)
\]

where \( \hat{n} \) stands for anti-normal order.

2.5.4. Lemma 2

Let us consider again the function \( f(\hat{n}) = \exp(-\gamma \hat{n}) \). This gives us that \( f(x) = e^{-\gamma x} \) and \( f(-u) = e^{\gamma u} \). Therefore

\[
\Delta^m f(u = 0) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} e^{\gamma k} = (e^{\gamma} - 1)^m,
\]

such that we can obtain the exponential of the number operator in anti-normal order (lemma) as

\[
e^{-\gamma \hat{n}} = e^{\gamma} e^{(1-e^{\gamma})\hat{n}}.
\]
2.5.5. Coherent states.

Let us use Eq. (89) to find averages for coherent states, \( |\alpha\rangle = \hat{D}(\alpha)|0\rangle \), where \( \hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \) is the so-called displacement operator and \( |0\rangle \) is the vacuum state:

\[
\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^\gamma \langle \alpha | \sum_{m=0}^{\infty} \frac{(1 - e^\gamma)^m}{m!} \hat{a}^m [\hat{a}^\dagger]^m | \alpha \rangle ,
\]

(90)

by using that

\[
\langle \alpha | \hat{a}^m [\hat{a}^\dagger]^m | \alpha \rangle = \langle 0 | (\hat{a} + \alpha)^m (\hat{a}^\dagger + \alpha^*)^m | 0 \rangle 
= \sum_{k=0}^{m} |\alpha|^{2k} \left( \frac{m!}{(m-k)!k!} \right)^2 (m-k)!,
\]

(91)

we may write

\[
\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^\gamma \sum_{m=0}^{\infty} (1 - e^\gamma)^m L_m (-|\alpha|^2),
\]

(92)

where \( L_m(x) \) are the Laguerre polynomials of order \( m \). We can finally write a closed expression for the sum above [10] to obtain the expected result for coherent states

\[
\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^{\gamma |\alpha|^2 (e^{-\gamma}-1)}.
\]

(93)

2.5.6. Fock states.

For Fock or number states we obtain

\[
\langle n | e^{-\gamma \hat{n}} | n \rangle = e^\gamma \langle n | \sum_{m=0}^{\infty} \frac{(1 - e^\gamma)^m}{m!} \hat{a}^m [\hat{a}^\dagger]^m | n \rangle = e^\gamma \sum_{m=0}^{\infty} \frac{(1 - e^\gamma)^m (m + n)!}{m! n!}
\]

(94)

rearranging the sum above with \( k = n + m \) we have

\[
\langle n | e^{-\gamma \hat{n}} | n \rangle = e^\gamma \sum_{k=n}^{\infty} (1 - e^\gamma)^{k-n} \frac{k!}{n!(k-n)!},
\]

(95)

which has a closed expression, as \( \sum_{k=n}^{\infty} x^{k-n} \frac{k!}{n!(k-n)!} = (1 - x)^{-n-1} \) [8]

\[
\langle n | e^{-\gamma \hat{n}} | n \rangle = e^{-\gamma n}.
\]

(96)

3. Quasiprobability distribution functions: Wigner function

A classical probability density may be written as an integral of delta functions,

\[
P(q, p) = \int \delta(q - Q) \delta(p - P) P(Q, P) dQdP.
\]

(97)

Using the integral forms delta functions have, we may rewrite them as

\[
P(q, p) = \frac{1}{4\pi^2} \int e^{-iup(P-P')} e^{i(q-Q)P} P(Q, P) dudv dP
\]

or

\[
P(q, p) = \frac{1}{4\pi} \int e^{-iup} e^{iv} \left( \int P(Q, P) e^{iup} e^{-iQv} dQdP \right) dudv
\]

(98)

that is nothing but the Fourier transform of the average of the function \( e^{iup' - iqv'} \).
By applying the correspondence principle and recalling that averages in quantum mechanics are obtained in the form
\[ \langle e^{iup - ivq} \rangle = \text{Tr}\{e^{iup - ivq}\}. \]  
(100)
The trace may be realized in several basis: Fock states, coherent states, phase states, position eigenstates, etc. We choose the last one
\[ \text{Tr}\{e^{iup - ivq}\} = \int dq \langle q | e^{iup - ivq} | q \rangle, \]  
(101)
that by using Baker-Hausdorff theorem may be re-written as
\[ \text{Tr}\{e^{iup - ivq}\} = e^{iuv/2} \int dq e^{-ivq} \langle q | e^{iu/2} \rho | q - u/2 \rangle, \]  
(102)
by doing \( q = x + u/2 \)
\[ \text{Tr}\{e^{iup - ivq}\} = \int dx e^{-ivx} \langle x + u/2 | e^{-iu/2} \rho | x - u/2 \rangle, \]  
(103)
introducing this expression into the quantum mechanical version of (99) and integrating in \( v \)
\[ P(q, p) \equiv W(q, p) = \frac{1}{2\pi} \int du e^{-iuq} \langle q, p | e^{-iu/2} \rho | q, p + u/2 \rangle, \]  
(104)
In 1932, Wigner introduced this function \( W(q, p) \), known now as the his distribution function [11]. It contains complete information about the state of the system, \( |\psi\rangle \). From Eq. (99) we see that it may be written also as
\[ W(\alpha) = \frac{1}{\pi^2} \int \exp(\alpha \beta^* - \alpha^* \beta) C(\beta) d^2 \beta, \]  
(105)
where \( C(\beta) \) in terms of annihilation and creation operators is given by
\[ C(\beta) = \text{Tr}\{\hat{\rho} \exp(\beta \hat{a}^\dagger - \beta^* \hat{a})\}, \]  
(106)
that receives the name of characteristic function. In Figs. 3-5 we plot the Wigner function for some states of harmonic oscillator: number states (Fig. 3), coherent states (Fig. 4) and displaced number states (Fig. 5).

3.1. Properties of the Wigner function

It is easy to see that, if we integrate in \( p \) we obtain
\[ \int W(q, p) dp = \frac{1}{2\pi} \int dp e^{-iuq} \langle q - u/2 | \rho | q + u/2 \rangle = \int \langle q - u/2 | \rho | q + u/2 \rangle \delta(u) du, \]  
(107)
so that
\[ \int W(q, p) dp = P(q). \]  
(108)
This is, the integral on momentum gives the marginal probability as a function of position. Now we see that doing the opposite, i.e. integrating in position gives the marginal probability in momentum space. However, to integrate the Wigner function in \( q \) is not so direct, therefore we do a similar analysis as the one we did to obtain the Wigner function, but with the probability as a function only of \( p \) instead of the combined probability, we have
\[ P(p) = \frac{1}{2\pi} \int e^{iup} \text{Tr}\{\rho e^{-iup}\} du = \frac{1}{2\pi} \int e^{iup} \langle q | e^{-iu/2} \rho e^{-iu/2} | q \rangle dq \]  
(109)
that just as we did before may be re-written as
\[ P(p) = \frac{1}{2\pi} \int e^{iup} \langle q + u/2 | \rho | q - u/2 \rangle dq \]  
(110)
that is nothing but the Wigner function integrated in position.
3.2. Obtaining the trace from the Wigner function

We can generalize the Wigner function for the density operator to any given operator

\[ W_\phi(q,p) = \frac{1}{2\pi} \int \! du e^{iuq} \langle q + \frac{u}{2} \phi | q - \frac{u}{2} \rangle. \]  

(111)

We can find an overlap formula between the density matrix and the operator \( \phi \) as

\[ \text{Tr}\{ \hat{\rho} \hat{\phi} \} = \int dq dp W(q,p) W_\phi(q,p). \]  

(112)

This result may be easily shown in the following way

\[ W(q,p)W_\phi(q,p) = \frac{1}{(2\pi)^2} \int \! dx_1 dx_2 e^{i(x_1 + x_2)q} q - \frac{x_1}{2} | \hat{\rho} | q + \frac{x_1}{2} \rangle \langle q - \frac{x_2}{2} | \hat{\phi} | q + \frac{x_2}{2} \rangle, \]  

(113)

the double integral on \( q \) and \( p \) of the above product of Wigner functions is

\[ \int \! W(q,p)W_\phi(q,p) dq dp = \frac{1}{(2\pi)^2} \int \! dq dp \]  

(114)

\[ \times \int \! dx_1 dx_2 e^{i(x_1 + x_2)q} q - \frac{x_1}{2} | \hat{\rho} | q + \frac{x_1}{2} \rangle \langle q - \frac{x_2}{2} | \hat{\phi} | q + \frac{x_2}{2} \rangle. \]  

(115)

integrating in \( p \) we obtain a delta function \( \delta(x + x') \) that may be readily integrated in \( x' \) to yield

\[ \int \! W(q,p)W_\phi(q,p) dq dp = \int \! dq dx_1 q - \frac{x_1}{2} | \hat{\rho} | q + \frac{x_1}{2} \rangle \langle q + \frac{x_2}{2} | \hat{\phi} | q - \frac{x_2}{2} \rangle, \]  

(116)

3.3. Symmetric averages

If we consider the characteristic function (106) function

\[ C(\lambda \beta) = \text{Tr}\{ \hat{\rho} \exp(\lambda [\beta \hat{a}^\dagger - \beta^* \hat{a}]) \}, \]  

(117)

differentiating it with respect to \( \lambda \) and evaluating at \( \lambda = 0 \), we have

\[ \frac{dC(\lambda \beta)}{d\lambda} |_{\lambda=0} = \text{Tr}\{ \hat{\rho} (\beta \hat{a}^\dagger - \beta^* \hat{a}) \}. \]  

(118)

By repeating the procedure \( k \) times we have

\[ \frac{d^k C(\lambda \beta)}{d\lambda^k} |_{\lambda=0} = \text{Tr}\{ \hat{\rho} (\beta \hat{a}^\dagger - \beta^* \hat{a})^k \}. \]  

(119)

On the other hand, from (105) we can write the characteristic function as the Fourier transform of the Wigner function

\[ C(\lambda \beta) = \frac{1}{\pi^2} \int \! \exp[\lambda(\alpha^* \beta \alpha - \beta^*)] W(\alpha) d^2 \alpha, \]  

(120)

by differentiating the above expression \( k \) times and evaluating at \( \lambda = 0 \) we obtain

\[ \frac{d^k C(\lambda \beta)}{d\lambda^k} |_{\lambda=0} = \frac{1}{\pi^2} \int \! (\alpha^* \beta \alpha - \beta^*)^k W(\alpha) d^2 \alpha. \]  

(121)

Equating (121) with (119) we can see that the Wigner function may be used to obtain averages of symmetric function of creation and annihilation operators

\[ \frac{1}{\pi^2} \int \! (\alpha^* \beta \alpha - \beta^*)^k W(\alpha) d^2 \alpha = \text{Tr}\{ \hat{\rho} (\beta \hat{a}^\dagger - \beta^* \hat{a})^k \}. \]  

(122)
3.4. Series representation of the Wigner function

We can obtain a series representation (non-integral) of the Wigner function by making \( y = u / 2 \) and inserting it into equation (104) we obtain

\[
W(q, p) = \frac{1}{\pi} \int dy \langle -y | e^{-i q \hat{p}} \rho e^{i q \hat{p}} e^{-i p \hat{q}} | y \rangle e^{-2 i y p},
\]

(123)

we can further put the exponential term in the integral above inside the bracket

\[
W(q, p) = \frac{1}{\pi} \int dy \langle -y | e^{-i p \hat{q}} e^{-i q \hat{p}} \rho e^{i q \hat{p}} e^{-i p \hat{q}} | y \rangle,
\]

(124)

and by using the parity operator \( \hat{\Pi} = (-1)^k \) we obtain

\[
W(q, p) = \frac{1}{\pi} \int dy \langle y | \hat{\Pi} e^{-i p \hat{q}} e^{-i q \hat{p}} \rho e^{i q \hat{p}} e^{-i p \hat{q}} | y \rangle,
\]

(125)

that is nothing but the trace of the operator

\[
\frac{1}{\pi} (-1)^k \hat{D}^\dagger(\alpha) \rho \hat{D}(\alpha),
\]

(126)

and which may be done in any basis, in particular we can use the Fock basis, to finally obtain

\[
W(q, p) = \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle k | \hat{D}^\dagger(\alpha) \rho \hat{D}(\alpha) | k \rangle.
\]

(127)

Recall that the states \( \hat{D}(\alpha) | k \rangle \) are the so-called displaced number states introduced in Section II. The Wigner function written as a series representation can be viewed in two ways, as a (weighted) sum of expectation values in terms of displaced number states, and as (weighted) sum of expectation values of a displaced density matrix.

3.5. Glauber-Sudarshan \( P \)-function

Although coherent states form an overcomplete basis, they may be used to represent states of the harmonic oscillator. If we use

\[
\hat{\rho} = \frac{1}{\pi^2} \int \int \langle \alpha | \hat{\rho} | \beta \rangle | \alpha \rangle \langle \beta \rangle d^2 \alpha d^2 \beta
\]

(128)

this representation involves off-diagonal elements \( \langle \alpha | \hat{\rho} | \beta \rangle \), and two integrations in phase space. The next diagonal representation was introduced independently by Glauber and Sudarshan [2]

\[
\hat{\rho} = \int P(\alpha) | \alpha \rangle \langle \alpha | d^2 \alpha
\]

(129)

which involves only one integration.

\[
P(\alpha) = \frac{1}{\pi^2} \int \exp(\alpha^* \beta - \alpha \beta^*) Tr\{ \hat{\rho} \exp(\beta \hat{a}^\dagger) \exp(-\beta^* \hat{a}) \} d^2 \beta.
\]

(130)

From Eq. (129) we can see that

\[
\langle [\hat{a}^\dagger]^n \hat{a}^k \rangle = Tr\{ [\hat{a}^\dagger]^n \hat{a}^k \hat{\rho} \} = \int P(\alpha) Tr\{ [\hat{a}^\dagger]^n \hat{a}^k | \alpha \rangle \langle \alpha | \} d^2 \alpha
\]

\[
= \int P(\alpha) Tr\{ \hat{a}^k | \alpha \rangle \langle \alpha | [\hat{a}^\dagger]^n \} d^2 \alpha
\]

\[
= \int P(\alpha) \alpha^k [\alpha^*]^n d^2 \alpha
\]

(131)

that indicates that this function may be used to calculate averages of normally ordered products of creation and annihilation operators.
3.6. Q or Husimi function

The Q or Husimi function, which is expressed as the coherent state expectation value of the density operator

\[ Q(\alpha) = \frac{1}{\pi^2} \int \exp(\alpha \beta^* - \alpha^* \beta) \text{Tr} \{ \hat{\rho} \exp(-\beta^* \hat{a}) \exp(\beta \hat{a}^\dagger) \} d^2 \beta, \tag{132} \]

the alternative form is

\[ Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle, \tag{133} \]

because \( \hat{\rho} \) is a positive operator \( Q(\alpha) \geq 0 \). Let us note that

\[ \int Q(\alpha) e^{\alpha \beta^* - \alpha^* \beta} d^2 \beta = \frac{1}{\pi} \text{Tr} \{ \hat{\rho} \} = 1 \]

\[ \int Q(\alpha) e^{\alpha \beta^* - \alpha^* \beta} d^2 \beta = \frac{1}{\pi} \text{Tr} \{ \hat{\rho} \} = 1 \]

this is, the Q-function may be used to calculate averages of creation and annihilation operators in anti-normal order.

3.7. Relations between quasiprobabilities

3.7.1. Differential forms

It is possible two group the Wigner, the Glauber-Sudarshan and the Husimi functions in a parametric form:

\[ F(\alpha, s) = \frac{1}{\pi} \int C(\beta, s) \exp(\alpha \beta^* - \alpha^* \beta) d^2 \beta \tag{136} \]

where \( C(\beta, s) \) is the characteristic function of order \( s \)

\[ C(\beta, s) = \text{Tr} \{ \hat{D}(\beta) \hat{\rho} \} \exp(s |\beta|^2 / 2) \tag{137} \]

with \( s \) a parameter that defines which is the function we are looking at. For \( s = 1 \) it is obtained the P-function, for \( s = 0 \) the Wigner function, and for \( s = -1 \) the Q-function.

The Q-function is then

\[ Q(\alpha) = \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) d^2 \beta \tag{138} \]

and for \( s = 0 \) the Wigner function

\[ W(\alpha) = \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) \exp(|\beta|^2 / 2) d^2 \beta \tag{139} \]

where

\[ G(\beta) = \frac{1}{\pi^2} \text{Tr} \{ \hat{D}(\beta) \hat{\rho} \} \exp(- |\beta|^2 / 2), \tag{140} \]

The equation above may be written as an infinite series and inserted into (139) to obtain

\[ W(\alpha) = \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) |\beta|^{2n} d^2 \beta. \tag{141} \]
Considering the equality
\[
\frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*} \exp(\alpha \beta^* - \alpha^* \beta) = -|\beta|^2 \exp(\alpha \beta^* - \alpha^* \beta)
\] (142)
we can cast equation (141) into
\[
W(\alpha) = \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \left(-\frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*}\right)^n Q(\alpha),
\] (143)
or, finally
\[
W(\alpha) = \exp \left(-\frac{1}{2} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*}\right) Q(\alpha).
\] (144)

3.7.2. Integral forms

We can obtain the \(Q\)-function from the \(P\)-function via (129) as
\[
Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle = \frac{1}{\pi} \int P(\beta) \langle \alpha | \beta \rangle \langle \beta | \alpha \rangle d^2 \beta,
\] (145)
i.e. we can finally write
\[
Q(\alpha) = \frac{1}{\pi} \int P(\beta)e^{-|\beta-\alpha|^2} d^2 \beta.
\] (146)
A similar relation may be obtained for the Wigner and \(Q\)-functions
\[
Q(\alpha) = \frac{1}{\pi} \int W(\beta)e^{-2|\beta-\alpha|^2} d^2 \beta.
\] (147)

3.8. Wigner function as a tool to calculate divergent (or not) series

One can use the Wigner function in the series representation to find the Wigner function associated with any operator (here we use a different normalization of the Wigner function)
\[
W_{\hat{\phi}}(q,p) = 2 \sum_{k=0}^{\infty} (-1)^k \langle k | \hat{D}^\dagger (\alpha) \hat{\phi} \hat{D}(\alpha) | k \rangle
\] (148)
Note that if \(\hat{\phi} = \hat{X} = \frac{\hat{a} + \hat{a}^*}{\sqrt{2}}\) one obtains
\[
q = W_{\hat{X}}(q,p) = 2 \sum_{k=0}^{\infty} (-1)^k \langle k | (\hat{X} + \frac{\hat{a} + \hat{a}^*}{\sqrt{2}}) | k \rangle = 2\frac{\alpha + \alpha^*}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k
\] (149)
because \(\alpha = (q + ip)/\sqrt{2}\)
\[
W_{\hat{X}}(q,p) = 2q \sum_{k=0}^{\infty} (-1)^k
\] (150)
this shows that \(\sum_{k=0}^{\infty} (-1)^k = 1/2\). Inserting the number operator in (1) we obtain
\[
W_n(q,p) = 2 \sum_{k=0}^{\infty} (-1)^k (k + |\alpha|^2)
\] (151)
or
\[
W_n(q,p) = \frac{q^2 + p^2}{2} + 2 \sum_{k=1}^{\infty} (-1)^k k.
\] (152)
On the one hand we have that
\[
\int W(q,p)W_\alpha(q,p)dqdp = Tr[\hat{n}\rho] = \langle \hat{n} \rangle
\]
and on the other hand we know that we can calculate averages of symmetric forms of creation and annihilation operators
\[
\langle \hat{n} \rangle = \int W(\alpha)\frac{\alpha\alpha^* + \alpha^*\alpha}{2}d^2\alpha - \frac{1}{2} = \int W(\alpha)|\alpha|^2d^2\alpha - \frac{1}{2}
\]
therefore
\[
W_\alpha(q,p) = \frac{q^2 + p^2}{2} - \frac{1}{2} \quad \text{or} \quad W_\alpha(q,p) + \frac{1}{2} = \frac{q^2 + p^2}{2}
\]
i.e. we can obtain the value for the (non-convergent) sum
\[
\sum_{k=1}^{\infty} (-1)^k k = -\frac{1}{4}.
\]
The results above show that the Wigner function may be used to calculate infinite series[12].

### 3.9. Number-phase Wigner function

In analogy to equation (106) we can define the function
\[
\tilde{C}_{\hat{n} - \hat{\phi}}(k,\theta) = \frac{1}{2}Tr\left[\left(D_{\hat{n} - \hat{\phi}}(k,\theta)e^{-i(k\phi - n\theta)} + c.c.\right)\hat{\rho}\right],
\]
where
\[
D_{\hat{n} - \hat{\phi}}(k,\theta) = e^{i\theta k}e^{-i\hat{\phi}k}(\hat{V}^\dagger)^k,
\]
with \(\hat{V}^\dagger = \sum_{k=0}^{\infty} |k+1\rangle\langle k|\) the Susskind-Glogower operator [4]. Because the Susskind-Glogower formalism fails in the phase description of the electromagnetic field with small photon numbers the unitarity of \(\hat{V}\) is spoiled. Also the fact that there is not a well defined phase operator, one can not use an expression of the form \(\exp[i(k\hat{\phi} - \hat{\phi}\hat{n})]\), and we use instead a "factorized" form in Eq. (158).

Note that in order to produce a real Wigner function we added the complex conjugate in (157) (because \(n\) can not be a negative integer). Eq. (105) does not have this problem because the integrations over \(\beta_x\) and \(\beta_p\) are from \(-\infty\) to \(\infty\). By writing the density matrix in the number state basis,
\[
\hat{\rho} = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} Q_{m,l}|m\rangle\langle l|,
\]
we obtain
\[
\tilde{C}_{\hat{n} - \hat{\phi}}(k,\theta) = \frac{e^{i\theta k}}{2} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} Q_{m,l}Tr[\hat{V}^\dagger]^ke^{-i\hat{\phi}k}|m\rangle\langle l|]e^{-i(k\phi - n\theta)} + c.c..
\]
The double integration over the whole phase space in (105) becomes here a sum and a single integration
\[
W(n,\phi) = \frac{1}{(2\pi)^2} \sum_{k=-n}^{\infty} \int_0^{2\pi} \tilde{C}_{\hat{n} - \hat{\phi}}(k,\theta)d\theta.
\]
Inserting equation (160) into (161) we obtain
\[
W(n,\phi) = \frac{1}{4\pi} \sum_{k=-n}^{\infty} \left( Q_{n,n+k}e^{-ik\phi} + Q_{n+k,n}e^{ik\phi} \right).
\]
It is easy to show that integrating (162) over the phase \(\phi\)
\[
\int_0^{2\pi} W(n,\phi)d\phi = Q_{n,n} = P(n),
\]
gives the photon distribution. And adding (162) over \(n\)
\[
P(\phi) = \sum_{n=0}^{\infty} W(n,\phi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_{n,m}e^{-i(m-n)\phi}
\]
produces the correct phase distribution. It is worth to note that for a number state \(|M\rangle\) equation (162) reduces to \(W(n,\phi) = \delta_{nM}/2\pi\), i.e. it is different from zero only for \(n = M\) as it should be expected.
3.9.1. Coherent state

The phase-number Wigner function for a coherent state

\[ |\alpha \rangle = e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \]  

(165)

is given by

\[ W(n,\phi) = \frac{e^{-|\alpha|^2 \alpha^n}}{2\pi \sqrt{n!}} \sum_{k=0}^{\infty} \frac{\alpha^k \cos[(n-k)\phi]}{\sqrt{k!}}. \]  

(166)

In Fig. 6 it is plotted the phase-number Wigner function for an amplitude \( \alpha = 4 \) and \( \phi = 0.5 \). Besides being always positive, it may be noted a smooth behavior. It may be seen a unique contribution of the single coherent state localized at the phase value \( \phi = 0.5 \).

3.9.2. A special superposition of number states

Let us consider the state

\[ |\phi_M \rangle = \frac{1}{\sqrt{M+1}} \sum_{m=0}^{M} e^{im\phi_0} |m\rangle. \]  

(167)

This state tends to have a completely well defined phase as \( M \) tends to infinity. For this state the phase-number Wigner function reads

\[ W(n,\phi) = \frac{1}{2(M+1)\pi} \sum_{k=0}^{M} \cos[(n-k)(\phi - \phi_0)], \]  

(168)

that may be put in the form Grad

\[ W(n,\phi) = \frac{1}{2(M+1)\pi} \cos \left( \frac{M}{2} - n \right) (\phi - \phi_0) \sin \left( \frac{M + 1}{2} (\phi - \phi_0) \right) \csc \left( \frac{\phi - \phi_0}{2} \right) \]  

(169)

In Fig. 7 it is plotted (169) for \( M = 20 \) and \( \phi_0 = 0.7 \). It shows a well defined phase or phase localization as it would be expected for a state of the form (167). It is also seen that as \( \phi \) approaches the value \( \phi_0 \) the maximum value for the phase-number Wigner function for the state (167) is obtained. From (169) it may be shown that this value is \( 1/2\pi \). By adding over \( n \) equation (169) the phase distribution is obtained

\[ P(\phi) = \frac{1}{2(M+1)\pi} \sin^2 \left( \frac{M + 1}{2} (\phi - \phi_0) \right) \csc^2 \left( \frac{\phi - \phi_0}{2} \right) \]  

(170)

that corresponds to the phase distribution for the state (167).

3.10. Wigner function in classical optics

Although we have shown that the Wigner function is a quantum object, in the sense that we have to make use of the properties of position and momentum operators, it may be realized in classical optics. Here we present a single scheme to generate it in the laboratory. First note that we need a one-dimensional Fourier transform of a displaced object in expression (104). The 1-D Fourier transform may be realized using the setup of Fig. 8 by placing the displaced object, Fig. 9 on the first plane. The cylindrical lens produces the 1-D Fourier transform of this object and in the second plane we can take a photograph of the squared Wigner function. This photograph as well as a plot of the squared Wigner function produced in the computer is shown in Fig 10.

4. More states of the field

In the first Section we introduced some states of the harmonic oscillator, such as Fock, coherent, displaced and phase states. In this Section we introduce some other states of the electromagnetic field.
4.1. Squeezed states

Squeezed states may be obtained by the application of a unitary squeeze operator defined as

\[ \hat{S}(r) = \exp \left( r \frac{\hat{a}^2 - \hat{a}^\dagger2}{2} \right) \]  

(171)

where \( r \) is an arbitrary (for simplicity we take it real, but it may be complex) number. The squeeze operator has the properties

\[ \hat{S}^\dagger(r) = \hat{S}^{-1}(r) = \hat{S}(-r), \]  

(172)

and transforms annihilation and creation operators as

\[ \hat{S}^\dagger \hat{a} \hat{S}(r) = \mu \hat{a} + \nu \hat{a}^\dagger, \quad \hat{S}^\dagger \hat{a}^\dagger \hat{S}(r) = \mu \hat{a}^\dagger - \nu \hat{a} \]  

(173)

with \( \mu = \cosh r \) and \( \nu = \sinh r \). A squeezed state is written as

\[ |\alpha, r\rangle = \hat{S}(r) |\alpha\rangle. \]  

(174)

The photon distribution for the squeezed states is given by

\[ P(n) = \frac{1}{\mu n!} \left( \frac{\nu}{2\mu} \right)^n e^{-|\beta|^2 + \frac{\nu^2}{4\mu} |\alpha|^2} \left| H_n \left( \frac{\beta}{\sqrt{2\mu\nu}} \right) \right|^2 \]  

(175)

with \( \beta = \alpha(\mu + \nu) \). This distribution is plotted in Fig. 11 for two different values of \( r \). Extra distributions after the main distribution may be observed. The extra distributions may in principle be measured by passing atoms through a cavity that contains this squeezed quantized field. After the atom interacts with the squeezed state, atomic states (excited or ground) may be measured. The atomic inversions that are produced by such measurements will present features related to this extra distribution that are called *ringing revivals* [14].

The uncertainties for \( \hat{q} \) and \( \hat{p} \) are

\[ \langle \Delta \hat{q} \rangle = e^{-2r}, \quad \langle \Delta \hat{p} \rangle = e^{2r} \]  

(176)

where we have taken \( \omega = \hbar = 1 \). The decrease in position uncertainty for positive \( r \) may be seen in Fig. 12 where we plot the Wigner function.

4.2. Schrödinger cat states

Because coherent states are quasi-classical states, the superposition of two of them is the closest we can get to the paradox proposed by Schrödinger: the superposition of two classical states (cat dead plus cat alive). Therefore the state

\[ |\psi_{\text{cat}}^\pm\rangle = \frac{1}{\sqrt{2N_\pm}} (|\alpha\rangle \pm |\alpha\rangle) \]  

(177)

where \( N_\pm = 1/\sqrt{2([1 + \exp(-2|\alpha|^2])} \) is a normalization constant. In Fig. 13 we plot the photon distribution for the ”plus” cat with \( \alpha = 4 \), we see that only even photons are allowed in such a cat state. In case we had the ”minus” cat, only odd photon numbers would have non-zero probabilities.

In Fig. 14 we show the Wigner function for the ”plus” cat. It may be seen the usual characteristics of such a distribution for the cat state, namely, the two contributions of the coherent states at \( \alpha = 2 \) and \( \alpha = -2 \) and the quantum interference that produces oscillations in the quasiprobability distribution function (see also Fig. 10). Approximate cat states may be generated in several systems such as Kerr media [16], atom field interactions [17] and ion-laser interactions (see Section V).

4.3. Thermal distribution

Up to now we have been looking at states that may be called *pure* state, this is states that may be represented by a wave function. However, there are states that can not be represented by a wave function, but they have to be represented as an statistical mixture of density matrices. Those states are called mixed states. We consider now a thermal distribution which has a diagonal expansion in Fock states

\[ \hat{\rho}_{\text{Th}} = \sum_{n=0}^{\infty} \frac{\hat{n}^n}{1 + \hat{n}^{n+1}} |n\rangle \langle n|. \]  

(178)

with \( \hat{n} \) the average number of thermal photons. These states are known as thermal states. In Fig. 15 we plot the photon distribution for a thermal state.
5. CQED-like systems I: Ion-laser interaction

We now turn our attention to systems that may be treated similar to the atom field interaction, as they share the same kind of operator’s algebra. To start we study an ion interacting with a laser field. Consider the Hamiltonian of a single ion trapped in a harmonic potential in interaction with laser light in the optical rotating wave approximation [18]

\[ H = \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \omega_{21} \hat{A}_{22} + \hbar [g E^{(-)}(\hat{x}, t) \hat{A}_{12} + H.c.], \]  

(179)

where \( \hat{a} \) and \( \hat{A}_{ab} \) are the annihilation operator of a quantum of the ionic vibrational motion and the electronic (two-level) flip operator for the \( |b\rangle \rightarrow |a\rangle \) transition of frequency \( \omega_{21} \), respectively. \( \nu \) is the trap frequency, \( g \) the electronic coupling matrix element, and \( E^{(-)}(\hat{x}, t) \) the negative part of the classical electric field of the driving field.

We consider the electric field with frequency \( \omega = \omega_{21} + m \nu \) with \( m \) an integer number (that may take negative values)

\[ E^{(-)}(\hat{x}, t) = E_0 \exp(-i(\hbar \nu t - \omega_{21} + m \nu t)). \]  

(180)

If \( m \) is positive the laser field is said to be tuned to the \( m \)th lower sideband (see Fig. 16). Because considering integer \( m \)'s give rise to interesting effects, we will do so here. The position operator \( \hat{x} \) may be written as

\[ \hat{x} = \eta (\hat{a} + \hat{a}^\dagger), \]  

(181)

where \( k = 2\pi/\lambda \) is the wave vector of the driving field and

\[ \eta = 2\pi \sqrt{|0\rangle\langle 0|} / \lambda \]  

(182)

is the so called Lamb-Dicke parameter.

By inserting (180) into (179) we obtain the Hamiltonian

\[ \hat{H} = \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \omega_{21} \hat{A}_{22} + \hbar [\lambda E_0 \exp(-i[\eta (\hat{a} + \hat{a}^\dagger) - \omega_{21} + m \nu t]) \hat{A}_{12} + H.c.]. \]  

(183)

We can use the Baker-Hausdorff formula to factorize the exponential in the above equation

\[ \exp(-i[\eta (\hat{a} + \hat{a}^\dagger)]) = e^{-\frac{\eta^2}{2}} e^{-i\eta \hat{a}^\dagger} e^{-i\eta \hat{a}} \]  

\[ = e^{-\frac{\eta^2}{2}} \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-i\eta)^j s \hat{a}^j \hat{a}^s}{j! s!}. \]  

(184)

We obtain the interaction picture Hamiltonian by getting rid off the free terms in (183)

\[ \hat{H}_I = \hbar \Omega e^{-\frac{\eta^2}{2}} \left[ e^{-im\nu t} \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-i\eta)^j s \hat{a}^j \hat{a}^s}{j! s!} \right] \hat{A}_{12} + H.c. \]  

(185)

where \( \Omega = g E_0 \). Note that the above Hamiltonian contains terms that oscillate rapidly, namely, \( e^{-i\omega t (s+m-j)} \).

We can make use of the vibrational RWA, assuming that \( \Omega \ll \nu \), which leaves us only with one sum, as \( j = s + m \), i.e.

\[ \hat{H}_I = \hbar \Omega e^{-\frac{\eta^2}{2}} [(\eta)^m \sum_{s=0}^{\infty} \frac{(-\eta^2)^s \hat{a}^s}{(s+m)!} \hat{a}^s \hat{A}_{12} + H.c.], \]  

(186)

or

\[ \hat{H}_I = \hbar \Omega e^{-\frac{\eta^2}{2}} [(\eta \hat{a}^\dagger)^m \sum_{s=0}^{\infty} \frac{(-\eta^2)^s \hat{a}^s \hat{a}^s}{(s+m)!} \hat{A}_{12} + H.c.]. \]  

(187)

Recalling the notation introduced in Section 1, we can express the interaction Hamiltonian in terms of normal ordered Bessel functions

\[ \hat{H}_I = \hbar \Omega e^{-\frac{\eta^2}{2}} [(-i\eta \hat{a}^\dagger)^m : J_m(-\eta^2 \hat{n}) : \hat{A}_{12} + H.c.]. \]  

(188)

A more common way of expressing the interaction Hamiltonian (187) is to use Associated Laguerre polynomials. In order to find the expression we note that

\[ \hat{a}^s \hat{a}^* = \hat{a}^* \hat{a}^s \sum_{n=0}^{\infty} |n\rangle \langle n| = \frac{\hat{n}!}{(\hat{n} - s)!} \sum_{n=0}^{\infty} |n\rangle \langle n|, \]  

(189)

inserting this expression into (187) we obtain the interaction Hamiltonian as

\[ \hat{H}_I = \hbar \Omega e^{-\frac{\eta^2}{2}} [(-i\eta \hat{a}^\dagger)^m \hat{n}! \left( \frac{\hat{n}!}{(\hat{n} + m)!} \right)^{(m)} (\eta^2) \hat{A}_{12} + H.c.]. \]  

(190)
5.1. Adding vibrational quanta

From equation (187), we note that if the Lamb-Dicke parameter is much less than one, \( \eta \ll 1 \), we can remain to the lowest order in the sum, such that we obtain the so-called two-phonon Hamiltonian

\[
H_I = \hbar \epsilon [\hat{A}_{21}^2 + \hat{A}_{12}^2],
\]

with \( \epsilon = -\Omega/2 \). For the study of the dynamics of interest we need the time evolution described by the Hamiltonian (191). The advantage of the interactions of Jaynes-Cummings type consists in the fact that the Hamiltonian can easily be diagonalized, and the unitary evolution operator readily takes the form

\[
\hat{U}_I(t) = \sum_{m=0,1} |1, m\rangle \langle 1, m| \\
+ \sum_{n=0}^{\infty} \left[ \cos \left( \frac{\Omega_n t}{2} \right) (|1, n + 2\rangle \langle 1, n + 2| + |2, n\rangle \langle 2, n|) \\
- i \sin \left( \frac{\Omega_n t}{2} \right) (|1, n + 2\rangle \langle 2, n| + |2, n\rangle \langle 1, n + 2|) \right].
\]

The quantity \( \Omega_n \) is the two-photon Rabi frequency which is given by

\[
\Omega_n = 2\epsilon \sqrt{(n + 1)(n + 2)}.
\]

Using these results, the time evolution of the quantum state in the interaction picture is easily derived for arbitrarily chosen initial conditions,

\[
|\Psi(t)\rangle = \hat{U}_I(t)|\Psi(0)\rangle.
\]

If we consider as initial state the ion in its excited state \(|2\rangle\) and the vibrational state a coherent state, we can find the atomic inversion, this is the probability to find the ion in its excited state minus the probability to find it in the ground state

\[
W(t) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \cos(2\Omega_n t).
\]

We plot this function in Fig. 17 as a function of the scaled time \( \tau = \epsilon t \). The interaction gives rise to a quasi-regular evolution of the atomic inversion, unlike the case of one phonon resonance. This can be used for several purposes, among them, to add excitations to the vibrational state.

In Fig. 17 it can be seen that if the ion is in its excited state \(|2\rangle\), initially, after an interaction time \( \tau = \pi/\epsilon \) the ion ends up in its ground state \(|1\rangle\), giving all its energy to the vibrational state by adding two vibrational quanta.

Moreover, the effect of having the ion in its excited state and after an interaction time having it in the ground state is shared by all vibrational states, not only when it is prepared in a coherent state. We show this in Fig. 18 where we show the atomic inversion for a thermal distribution.

Let us consider again the case when we have initially the atom (before it enters the cavity) in its excited state and the cavity field is prepared in a coherent state \(|\alpha\rangle\), that is

\[
|\Psi(0)\rangle = |2\rangle |\alpha\rangle.
\]

Combining Eqs (192) and (196) and using a compact operator representation of the JC dynamics we arrive at

\[
|\Psi(t)\rangle = \cos \left( ct \sqrt{\hat{a}^2(\hat{a}^\dagger)^2} \right) |\alpha\rangle |2\rangle \\
- i(\hat{V}^\dagger)^2 \sin \left( ct \sqrt{\hat{a}^2(\hat{a}^\dagger)^2} \right) |\alpha\rangle |1\rangle.
\]

In order to derive illustrative analytical results, in the following we will apply the approximation

\[
\sqrt{\hat{a}^2(\hat{a}^\dagger)^2} \approx \hat{n} + \frac{3}{2}.
\]
Although it represents a Taylor-series expansion for large eigenvalues \( n \) of the operator \( \hat{n} \), the error is small already for small \( n \)-values. For example, for \( n = 1 \) the relative error is only 0.02. Based on this approximation one may simplify Eq. (197) as

\[
|\Psi(t)\rangle \approx \cos[\lambda t(\hat{n} + 3/2)]|\alpha\rangle|2\rangle - i(\hat{V}^\dagger)^2 \sin[\lambda t(\hat{n} + 3/2)]|\alpha\rangle|1\rangle.
\]  

(199)

Choosing a particular interaction time \( t = \tau \) according to

\[
\tau = \pi/\lambda,
\]

(200)

we obtain for the vibrational state vector

\[
|\Psi_+^{(1)}(\tau)\rangle_v \approx i(\hat{V}^\dagger)^2|\alpha\rangle - (\hat{V}^\dagger)^4|\alpha\rangle.
\]

(201)

By repeating the process \( k \) times one finally obtains for the quantum state, at the time \( t_k \) after completing \( k \) interactions,

\[
|\Psi_+^{(k)}(t_k)\rangle_v = i^k(\hat{V}^\dagger)^{2k}|\alpha\rangle.
\]

(202)

After many interactions, the state \( |\Psi_+^{(k)}(t_k)\rangle_v \) exhibits a strong sub-Poissonian character, because, while one is adding two excitations per interaction, at the same time one is keeping the width of the distribution constant.

The excitation distribution \( P_n^{(k)} \) after \( k \) interactions is easily found to be related to the number statistics \( P_n^{(0)} \) of the initial state \( |\alpha\rangle \) via

\[
P_n^{(k)} = P_{n-2k}^{(0)}.
\]

(204)

This result clearly shows that the number statistics is only shifted but retains its form. One way of studying the properties of the states being generated is through the Mandel \( Q \)-parameter [19]

\[
Q_M = \frac{\Delta n}{n} - 1,
\]

(205)

which reveals if a given photon distribution function is Poissonian, super-Poissonian or sub-Poissonian:

\[
\begin{align*}
Q > 0 & \quad \text{super-Poissonian} \\
Q = 0 & \quad \text{Poissonian} \\
Q < 0 & \quad \text{sub-Poissonian}
\end{align*}
\]

(206)

The Mandel \( Q \)-parameter is given by

\[
Q = \frac{|\alpha|^2}{|\alpha|^2 + 2k} - 1,
\]

(207)

and, as the number of interactions increases, the Mandel \( Q \)-parameter approaches the value \(-1\), i.e. the state acquires maximum sub-Poissonian character. The sub-Poissonian effect of the vibrational wave function becomes more significant with increasing number of interactions.

Moya-Cessa et al. [20] have shown also the possibility of subtracting excitations. This can be done, because, if instead of having the ion initially in the excited state \( |2\rangle \), we had it in the ground state \( |1\rangle \), after an interaction time \( \tau = \pi/\lambda \) the atomic inversion would be \(+1\), i.e. the ion would take the vibrational energy. This process repeated \( k \) times would produce a subtraction of \( 2k \) vibrational quanta.
5.2. Filtering specific superpositions of number states

If instead of one laser, we assume two lasers driving the ion, the first tuned to the jth lower sideband and the second to the mth lower sideband, we may write \( E(\cdot) \) as

\[
E(\cdot, t) = E_1 e^{-i(k_1x - \omega t + j\nu_1 t)} + E_m e^{-i(k_mx - \omega t + m\nu t)},
\]

where, if \( m = 0 \) it would correspond to the driving field being on resonance with the electronic transition, \( \hat{x} \) may be written as before

\[
k_s \hat{x} = \eta_s (\hat{a} + \hat{a}^\dagger),
\]

where \( \eta_s \) are the wave vectors of the driving fields and

\[
\eta_s = \frac{2\pi \sqrt{\langle 0 | \Delta x^2 | 0 \rangle}}{\lambda_s}
\]

are the LDPs with \( s = j, \ldots, m \).

In the resolved sideband limit, the vibrational frequency \( \nu \) is much larger than other characteristic frequencies and the interaction of the ion with the two lasers can be treated separately, using a nonlinear Hamiltonian [21]. The Hamiltonian (179) in the interaction picture can then be written as

\[
\hat{H}_I = \hbar \hat{A}_{21} \left[ \Omega_j e^{-\eta_j^2/2} \frac{\hat{n}!}{(\hat{n} + j)!} L_{\hat{n}}^{(j)}(\eta_j^2) \hat{a}^j + \Omega_m e^{-\eta_m^2/2} \frac{\hat{n}!}{(\hat{n} + m)!} L_{\hat{n}}^{(m)}(\eta_m^2) \hat{a}^m \right] + H.c.,
\]

(211)

where \( L_{\hat{n}}^{(k)}(\eta_k^2) \) are the operator-valued associated Laguerre polynomials and the \( \Omega_j, \Omega_m \) are the Rabi frequencies and \( \hat{n} = \hat{a}^\dagger \hat{a} \). The master equation which describes this system can be written as

\[
\frac{\partial \rho}{\partial t} = \frac{i}{\hbar} [\hat{H}_I, \rho] + \frac{\Gamma}{2} \left( 2\hat{A}_{12} \hat{\rho} \hat{A}_{21} - \hat{A}_{22} \hat{\rho} \hat{A}_{22}^\dagger - \hat{A}_{21} \hat{\rho} \hat{A}_{12}^\dagger \right)
\]

(212)

where the last term describes spontaneous emission with energy relaxation rate \( \Gamma \), and

\[
\hat{\rho} = \frac{1}{2} \int_0^1 ds W(s) e^{i\eta_E x} \hat{\rho} e^{-i\eta_E x},
\]

(213)

accounts for changes of the vibrational energy because of spontaneous emission. Here \( \eta_E \) is the LDP corresponding to the field (208) and \( W(s) \) is the angular distribution of spontaneous emission (see de Matos Filho and Vogel [21]).

The steady-state solution to Eq. (213) is obtained by setting \( \partial \hat{\rho} / \partial t = 0 \) and may be written as

\[
\hat{\rho}_s = |1 \rangle \langle \psi_s | \psi_s \rangle |1 \rangle,
\]

(214)

where \( |1 \rangle \) is the electronic ground state and \( |\psi_s \rangle \) is the vibrational steady-state of the ion, given by

\[
(\Omega_j e^{-\eta_j^2/2} \frac{\hat{n}!}{(\hat{n} + j)!} L_{\hat{n}}^{(j)}(\eta_j^2) \hat{a}^j + \Omega_m e^{-\eta_m^2/2} \frac{\hat{n}!}{(\hat{n} + m)!} L_{\hat{n}}^{(m)}(\eta_m^2) \hat{a}^m) \langle \psi_s | = 0.
\]

(215)

For simplicity, we will concentrate in the \( j = 1 \) and \( m = 0 \) case (single number state spacing) for which Eq. (214) is written as

\[
(\Omega_1 e^{-\eta_1^2/2} \frac{\hat{n}!}{\hat{n} + 1} L_{\hat{n}}^{(1)}(\eta_1^2) \hat{a} + \Omega_0 e^{-\eta_0^2/2} L_{\hat{n}}(\eta_0^2)) \langle \psi_s | = 0.
\]

(216)

Note that \( \hat{H}_I |1 \rangle \langle \psi_s | = 0 \) so that ion and laser have stopped to interact, which occurs when the ion stops to fluoresce. For the \( j = 1 \) and \( k = 0 \) case, and assuming \( L_{\hat{n}}^{(1)}(\eta_1^2) \neq 0 \) and \( L_k(\eta_0^2) \neq 0 \) for all \( k \), one generates nonlinear coherent states [22]. However, by setting a value to one of the LDPs such that, for instance, \( L_q(\eta_0^2) = 0 \),

(217)

for some integer \( q \), we obtain that, by writing \( |\psi_s \rangle \) in the number state representation,

\[
|\psi_s(\eta_0)\rangle = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} C_n^{(0)} |n\rangle,
\]

(218)

where

\[
n = 0, \ldots, \infty
\]

(219)
(the argument of $\psi_s$ denotes the condition we apply, i.e., in Eq (219), the condition is on $\eta_0$) where
\[ C_n^{(0)} = \left( \frac{\Omega_0 e^{-\eta_0^2/2}}{\Omega_1 e^{-\eta_1^2/2}} \right)^n (n!)^{1/2} \prod_{m=0}^{n-1} \frac{L_m(\eta_0^2)}{L_m(\eta_1^2)}, \quad C_0^{(0)} = 1, \tag{220} \]
and
\[ N_0^2 = \sum_{n=0}^{q} |C_n^{(0)}|^2 \tag{221} \]
is the normalization constant.

If, instead of condition (218), we choose
\[ L^{(1)}_p(\eta_1^2) = 0, \tag{222} \]
we obtain the wave function
\[ |\psi_s(\eta_1)\rangle = \frac{1}{N_1} \sum_{n=p+1}^{\infty} C_n^{(1)} |n\rangle, \tag{223} \]
where now,
\[ C_n^{(1)} = \left( \frac{\Omega_0 e^{-\eta_0^2/2}}{\Omega_1 e^{-\eta_1^2/2}} \right)^{n-p-1} \sqrt{\frac{n!}{(p+1)!}} \prod_{m=p+1}^{n-1} \frac{L_m(\eta_0^2)}{L_m(\eta_1^2)}, \quad C_{p+1}^{(1)} = 1, \tag{224} \]
and
\[ N_1^2 = \sum_{n=p+1}^{\infty} |C_n^{(1)}|^2. \tag{225} \]
Combining both conditions (218) and (222), one would obtain (for $q > p$)
\[ |\psi_s(\eta_0, \eta_1)\rangle = \frac{1}{N_{01}} \sum_{n=p+1}^{q} C_n^{(1)} |n\rangle, \tag{226} \]
with
\[ N_{01}^2 = \sum_{n=p+1}^{q} |C_n^{(1)}|^2. \tag{227} \]

In this way, by setting the conditions (218), (222) or both, we can engineer states in the following three zones of the Hilbert space, (a) from $|0\rangle$ to $|q\rangle$, (b) from $|p+1\rangle$ to $|\infty\rangle$, or (c) from $|p+1\rangle$ to $|q\rangle$. In the later case, by setting $q = p + 1$ generation of the number state $|q\rangle$ is achieved. This situation is better observed in Fig. 19, where may be seen the allowed zones to have excitations.

We should remark, that by selecting further apart sidebands one would obtain a different spacing in Eqs (219), (223) and (226). For instance, by choosing $j = 2$ and $k = 0$ one would obtain only even or odd number states in those equations (depending in this case on initial conditions and $W(s)$, the angular distribution of spontaneous emission). Also, it should be noticed that one can use the parameters $j = n + 1$ and $k = m$ (with $m \neq 0$) (in the single number state spacing case) to extend the possibilities of choosing LDPs. LDPs of the order of one (or less) are needed [for conditions (218) and (222)] which can be achieved by varying the geometry of the lasers. For example, by setting $\eta_0 = 1$, we have $L_1(\eta_0^2) = 1$, and therefore we obtain the qubit
\[ |\psi_s(\eta_0 = 1)\rangle = \frac{1}{\sqrt{1 + |\frac{\Omega_0 e^{-1/2}}{\Omega_1 e^{-\eta_1^2/2}}|}} \left( |0\rangle - \frac{\Omega_0 e^{-1/2}}{\Omega_1 e^{-\eta_1^2/2}} |1\rangle \right), \tag{228} \]
where by changing the Rabi frequencies, one has control of the amplitudes.

Finally, note that we could have also chosen to drive the $q$th upper sideband instead of the $k$th lower sideband in equation (208) with basically the same results.
6. CQED-like systems II: Mirror-field interaction

Consider a cavity with two perfectly reflecting mirrors, one of them fixed and the other one can move, undergoing harmonic oscillations. The cavity resonances are calculated in the absence of the impinging field. Given \( L \) to be the equilibrium cavity length the resonant angular frequencies of the cavity are

\[
\omega = k \pi \frac{c}{L}
\]  

(229)

where \( k \) is an arbitrary integer number and \( c \) the speed of light. We assume that the retardation effects due to the oscillating mirror may be neglected. We will use a field intensity such that the correction to the radiation pressure force, due to the Doppler frequency shift of the photons \([23]\) may also be neglected. Therefore we are able to write the relevant Hamiltonian as \([24]\)

\[
H = \hbar \omega a^\dagger a + \frac{p^2}{2m} + \frac{m\Omega^2 x^2}{2} + H_{\text{int}}
\]  

(230)

where \( a \) and \( a^\dagger \) are the annihilation and creation operators for the cavity field, respectively. The field frequency is \( \omega \), the mirror oscillates at a frequency \( \Omega \), \( p \) and \( x \) are the momentum and displacement from the equilibrium position operators of the oscillating mirror with mass \( m \) and \( H_{\text{int}} \) accounts for the interaction between the cavity mode and the oscillating mirror. Because we have assumed no retardation effects, we may simply write

\[
H_{\text{int}} = -\hbar g a^\dagger a x,
\]  

(231)

where the coupling constant is

\[
g = \frac{\omega}{L} \sqrt{\frac{\hbar}{2m\Omega}},
\]  

(232)

with \( m \) the mass of the movable mirror. \( H_{\text{int}} \) represents the effect of the radiation pressure force \( F_R = (\hbar \omega / L)a^\dagger a \) that causes the instantaneous displacement \( x \) of the mirror.

We can therefore rewrite the Hamiltonian (230) in the form

\[
H = \hbar (\omega a^\dagger a + \Omega b^\dagger b - g a^\dagger a(b^\dagger + b))
\]  

(233)

\( b \) and \( b^\dagger \) are the annihilation and creation operators for the mirror.

It is convenient to write the Hamiltonian (233) with the help of displacement operators

\[
H = D_m(\eta a^\dagger a) (\omega a^\dagger a + \Omega b^\dagger b - \epsilon(a^\dagger a)^2) D_m^\dagger(\eta a^\dagger a)
\]  

(234)

where \( \epsilon = g\eta \) with \( \eta = g/\Omega \) and the displacement operator is given by

\[
D_m(\beta) = e^{\beta b^\dagger - \beta^* b},
\]  

(235)

with \( N = a^\dagger a \). Then the unitary evolution operator is simply

\[
U(t) = e^{-i\frac{\hbar}{\omega} D_m(\eta N)} e^{-i(\omega N + \Omega b^\dagger b - \epsilon N^2)} D_m^\dagger(\eta N)
\]  

(236)

We will consider the initial state of the field to be in a coherent state

\[
|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
\]  

(237)

and the initial state of the mirror to be arbitrary and denoted by the density matrix \( \rho_m \), so that the total density matrix at time \( t \) is given by

\[
\rho(t) = U(t)|\alpha\rangle\langle\alpha| \otimes \rho_m U^\dagger(t).
\]  

(238)

Once having the evolved density matrix may calculate the average of any operator, \( A \) by the total trace:

\[
\langle A \rangle = Tr\{\rho(t)A\} = Tr\{|\alpha\rangle\langle\alpha| \otimes \rho_m U^\dagger(t)AU(t)\}
\]  

(239)

where we have substituted (238) into the above equation and have made use of the invariance under permutations of the trace.

We can now calculate \( \langle a \rangle \) in the form

\[
\langle a \rangle = \alpha e^{-i(\omega + \Omega)t} Tr\left[\rho_m D_m(\eta e^{i\Omega t}) D_m(-\eta) |\alpha e^{2i(\xi - \eta^2 \sin \Omega t)}\rangle\langle\alpha|\right]
\]  

(240)
where we have used several times the properties of permutation under the trace symbol. By using that
\[ D_m \left( \eta e^{i\Omega t} \right) D_m \left( -\eta e^{i\Omega t} \right) = D_m \left( \eta e^{i\Omega t} - 1 \right) \] we may finally write
\[ \langle a \rangle = a e^{-i(\omega+\epsilon)t} e^{-i\eta^2 \sin\Omega t} e^{-|\eta|^2 (t-\eta^2 \sin\Omega t)} \chi_m \left( \eta e^{i\Omega t} - 1 \right) \] where \( \chi_m \left( \eta e^{i\Omega t} - 1 \right) = Tr \{ \rho_m D_m [\eta e^{i\Omega t} - 1] \} \) is the characteristic function associated to the density matrix \( \rho_m \). Therefore, by measuring the quadratures of the field (see for instance Leonhardt [13]) \( \langle X \rangle = \langle (a + a^\dagger) \rangle / \sqrt{2} \) and \( \langle Y \rangle = -i \langle (a - a^\dagger) \rangle / \sqrt{2} \) we may obtain the average value for the annihilation operator and hence, information about the state of the mirror through its characteristic function. The argument of the characteristic function may be changed in some range of parameters as \( \omega \sim 10^{16} s^{-1}, \Omega \sim 1 \text{ kHz}, L \sim 1 \text{ m} \) and \( m \sim 10 \text{ mg} \) (see Mancini et al. [25]).

6.1. Wigner function in terms of characteristic function

We now write the characteristic function in terms of the average value of the annihilation operator
\[ \chi_m (\eta e^{i\Omega t} - 1) = \frac{\langle a \rangle}{\alpha e^{-i(\omega+\epsilon)t} e^{-i\eta^2 \sin\Omega t} e^{-|\eta|^2 (t-\eta^2 \sin\Omega t)}} \] from which we can obtain the set of \( s \)-parametrized quasiprobability distributions, and in particular the Wigner function for the mirror state
\[ W_m (\xi) = \frac{1}{\pi^2} \int d^2 \beta e^{\xi \beta^{*} - \xi^{*} \beta} \chi_m (\beta), \] where we have defined \( \beta = \eta e^{i\Omega t} - 1 \). Note that a value of \( \beta \) defines only one point in the dual phase space, or, in other words, a particular set of parameters, such as interaction time, mirror-field interaction constant, etc., define only one value of \( \beta \). The transformation of the characteristic function above requires an infinite set of points (in general, a continuous set of values from minus infinity to infinity. Therefore the precision of the method pointed out here is related with the amount of times the experiment has to be repeated. However, this is a ‘problem’ related to all reconstruction schemes.

What makes it possible to obtain information about the mirror state is the initial coherence of the field and the form of the Hamiltonian that has the term
\[ b + b^\dagger. \] Wiltens and Meystre [28] had shown that for the Jaynes-Cummings Model it was possible to obtain information about the characteristic function of the field only if the system interacted with an extra (classical) field to allow several absorptions \( \langle a^k \rangle \) or emissions \( \langle [a^\dagger]^k \rangle \) such that moments of \( a \) and/or \( a^\dagger \) could be obtained, i.e. the characteristic function reconstructed. Here, the multiple absorptions/emissions are given by the term in Eq. (245), being this term responsible for the possibility of the reconstruction.

7. Quantum phase

Classically we may decompose a complex number, \( A \), in amplitude and phase by simply writing \( A = re^{i\phi} \), with \( r = |A| \) and
\[ \phi = -i \ln \frac{A}{r}, \] where it is implied that we have chosen the principal branch of the multi-valued logarithm function.

A Hermitian operator in correspondence to the classical form (246) was proposed by Arroyo Carrasco and Moya-Cessa [7]
\[ \hat{\phi} = -\frac{i}{2} \lim_{\chi \to \infty} \hat{D}(\chi) \ln \left( 1 + \frac{\hat{a}}{\chi} \right) \hat{D}^\dagger(\chi) + h.c., \]
where \( \hat{a} \) and \( \hat{a}^\dagger \) are the annihilation and creation operator for the harmonic oscillator, respectively. \( D(\chi) = e^{\chi(\hat{a}^\dagger - \hat{a})} \) is the displacement operator with \( \chi \) a real parameter that tends to infinity to ensure convergence of the series

\[
\ln \left( 1 + \frac{\hat{a}}{\chi} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \frac{\hat{a}}{\chi} \right)^k.
\]  

(248)

Note that the displacement operators in Eq. (247) produce a displacement of \( \hat{a} \) by an amount minus \( \chi \) producing exactly the form (246). However we keep the displacement operator explicitly in order to have a Taylor series for the logarithm.

The operator (247) may be found to be Turski’s operator [29] if we use the unity operator given in terms of coherent states, \( \hat{1} = \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2 \alpha \), and insert it into (247) it yields

\[
\hat{\phi} = -\frac{i}{2} D(\chi) \ln \left( 1 + \frac{\hat{a}}{\chi} \right) \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2 \alpha \hat{D}(\chi) + h.c.,
\]

(249)

that may finally be written as

\[
\hat{\phi} = -\frac{i}{2\pi} \int (\ln \alpha - \ln \alpha^*)|\alpha\rangle \langle \alpha| d^2 \alpha.
\]

(250)

Again, choosing the principal branch in the above equation, we can rewrite (250) as

\[
\hat{\phi} = \frac{1}{\pi} \int \theta |\alpha\rangle \langle \alpha| d^2 \alpha = \hat{\phi} \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2 \alpha,
\]

(251)

where \( \theta = \text{arg}(\alpha) \). Of course, as any operator that lives in the whole Hilbert space, \( \hat{\phi} \) obeys the equation of motion

\[
\frac{d\hat{\phi}}{dt} = i\omega [\hat{a}^\dagger \hat{a}, \hat{\phi}]
\]

(252)

where \( \omega \) is the frequency of the harmonic oscillator. Note that for a phase operator defined in a finite dimensional Hilbert space to obey such equation of motion, the harmonic oscillator Hamiltonian should be defined also in a finite dimensional Hilbert space.

We can calculate the average value of the argument of the operator (251), given a wave function \( |\psi\rangle \), as

\[
\langle \hat{\phi} \rangle = \int \text{arg}(\alpha) Q(\alpha) d^2 \alpha,
\]

(253)

where

\[
Q(\alpha) = \frac{1}{\pi} |\langle \alpha|\psi\rangle|^2,
\]

(254)

is the Q-function.

### 7.1. A formalism for phase

A formalism for phase could be introduced based on (251) as follows

\[
\langle \text{arg}(\hat{a}) \rangle = \langle \hat{\phi} \rangle = \int \text{arg}(\alpha) Q(\alpha) d^2 \alpha,
\]

(255)

and

\[
\langle \text{arg}^k(\hat{a}) \rangle = \int \text{arg}^k(\alpha) Q(\alpha) d^2 \alpha.
\]

(256)

Note that there is no phase that can be defined in an strict correct form [5,7], therefore in the above we have used the following form:

\[
\hat{\phi} \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2 \alpha = \frac{1}{\pi} \int \theta |\alpha\rangle \langle \alpha| d^2 \alpha,
\]

(257)
and
\[ \hat{\phi}^n \frac{1}{\pi} \int |\alpha\rangle \langle \alpha| d^2 \alpha = \frac{1}{\pi} \int \theta^n |\alpha\rangle \langle \alpha| d^2 \alpha, \] (258)
such that we can calculate the phase variance, \( \Delta \phi \equiv \langle \arg(\hat{a}) \rangle - \langle \arg(\hat{a}) \rangle^2 \) for a number state \( |n\rangle \), yielding the result
\[ \Delta \phi = \frac{\pi^2}{3}, \] (259)
and where we have used
\[ Q(\alpha) = \frac{1}{\pi |\langle \alpha|n \rangle|^2} = e^{-|\alpha|^2 |\alpha|^{2n}} \] (260),
i.e. giving the correct phase variance expected for a state of undefined phase. Therefore, the operator given by Turski would lead to a phase formalism given by the (radially) integrated \( Q \)-function.

7.2. Coherent states

We can now calculate from (255) the phase properties for a coherent state
\[ |\beta\rangle = |\rho e^{i\eta}\rangle = e^{-\frac{\rho^2}{2}} \sum_{n=0}^{\infty} \frac{\rho^n e^{i\eta}}{\sqrt{n!}} |n\rangle, \] (261)
for which we obtain
\[ \langle \arg(\hat{a}) \rangle = \eta, \] (262)
we obtain the same result if we use (247):
\[ \langle \beta | \hat{\phi} | \beta \rangle = -\frac{i}{2} \langle \beta | \hat{D}(\chi) \ln \left( 1 + \frac{\hat{a}}{\chi} \right) \hat{D}^\dagger(\chi) | \beta \rangle + c.c. = \eta \] (263)
so that
\[ \Delta \phi_{coh} = \frac{\pi^2}{3} + 4e^{-\rho^2} \sum_{n>m}^{\infty} (-1)^{n-m} \frac{\rho^{n+m}}{n!m!} \frac{\Gamma(\frac{n+m}{2} + 1)}{(n-m)^2}, \] (264)
where \( \Gamma(x) \) is the well-known Gamma function.

Note that because the formalism that takes us to the integrated \( Q \)-function comes from the operator introduced by Turski. He showed that \( [n, \hat{\phi}] = i \), with \( \hat{n} = \hat{a}^\dagger \hat{a} \), leading to a Heisenberg uncertainty relation
\[ \Delta \phi \geq \frac{1}{2 \Delta n} = \frac{1}{2\rho^2}. \] (265)
This may be corroborated in Fig. 20 where we plot \( \Delta \phi_{coh} \) as a function of \( \rho \), together with the expression \( 1/2\rho^2 \).

7.3. Radially integrated Wigner function

In the former section Eqs. (251) and (253) were used to introduce the calculation of phase properties in terms of the \( Q \)-function, however one can also introduce them in terms of other quasiprobability distributions, namely, the Wigner function (see for instance Garraway and Knight [30])
\[ \langle \hat{\phi} \rangle_W = \int \arg(\alpha) W(\alpha) d^2 \alpha, \] (266)
and
\[ \langle \hat{\phi}^k \rangle_W = \int \arg^k(\alpha) W(\alpha) d^2 \alpha. \] (267)
Calculating phase fluctuations for the coherent state (261) using the above expressions leads to

\[
\Delta \phi_W = \frac{\pi^2}{3} + 4e^{-2\rho^2} \sum_{n>m}^{\infty} (-1)^{n-m} \frac{\sqrt{2} \rho^{n+m} \Gamma\left(\frac{n+m}{2} + 1\right)}{n!m!} \frac{1}{(n-m)^2},
\]

(268)

where we have used

\[
W(\alpha) = \frac{2}{\pi} e^{-2|\beta - \alpha|^2}.
\]

(269)

In Fig. 20 we plot \(\Delta \phi_W\) together with the expression for the phase variance for coherent states using the Pegg-Barnett formalism, that can be written as [31,32]

\[
\Delta \phi_{P-B}^{coh} = \frac{\pi^2}{3} + 4e^{-\rho^2} \sum_{n>m}^{\infty} (-1)^{n-m} \frac{\rho^{n+m}}{\sqrt{n!m!}} \frac{1}{(n-m)^2}.
\]

(270)

Note that the Heisenberg uncertainty relation in the Pegg-Barnett case leads to the inequality (for a coherent state) [33]

\[
\Delta \phi \geq \frac{1}{4\Delta n + 3/\pi^2} = \frac{1}{4\rho^2 + \pi^2/3}.
\]

(271)

This expression is also plotted in Fig. 21. It is seen that both expressions for the phase fluctuations, the one obtained from the Wigner function integration and the Pegg-Barnett formalism tend to the above limit.
References

[11] We could have done also the following: If we call $S = \sum_{k=1}^{\infty} (-1)^k k$ we have that $S = -1 + \sum_{k=2}^{\infty} (-1)^k k = -1 + \sum_{k=1}^{\infty} (-1)^{k+1}(k+1) = -1 - S + \sum_{k=1}^{\infty} (-1)^k$, and the last sum, from Eq. (2) is equal to 1/2.
**Figure 1** Coherent state with $\alpha = 6$. It may be observed that the distribution is centered in $\bar{n} = 36$ and has a width of approximately $2|\alpha|$.

**Figure 2** Photon number distribution for displaced number states for (a) $\alpha = 2$, $n = 20$, (b) $\alpha = 2$, $n = 20$ and (c) $\alpha = 5$, $n = 1$.

**Figure 3** Wigner function for the first number state $|1\rangle$.

**Figure 4** Wigner function for the coherent state of amplitude $|\alpha\rangle = 2$.

**Figure 5** Wigner function for the displaced number state state for the parameters $|\alpha\rangle = 4$ and $|n\rangle = 1$.

**Figure 6** Number phase Wigner function for a coherent state for an amplitude $\alpha = 4$ and $\phi = 0.5$.

**Figure 7** Number phase Wigner function for a superposition of the first twenty one Fock states.

**Figure 8** Setup to obtain the square of the Wigner function. An object (see next figure) is placed in the first plane and in the second plane is taken a photograph of the squared Wigner function (Fig. 10).

**Figure 9** The displaced object needed to obtain the Wigner function.

**Figure 10** Squared Wigner function of the displaced function, Fig. 9 (above). The computer-obtained squared Wigner function (below).

**Figure 11** Photon distribution for the squeezed state for $\alpha = 5$ and (a) $r = 1.5$ and (b) $r = 2$.

**Figure 12** Wigner function for the squeezed state for $\alpha = 2$ and $r = 2$. 
Figure 13  Photon number distribution for the cat state $|\psi_{\text{cat}}^+\rangle$, $\alpha = 4$.

Figure 14  Wigner function for the cat state $|\psi_{\text{cat}}^+\rangle$, $\alpha = 2$.

Figure 15  Photon distribution for the thermal distribution, with $\bar{n} = 3$.

Figure 16  Level scheme of a vibrating two level ion, with ground state $|1\rangle$ and excited state $|1\rangle$. There are shown several possible transitions when it interacts with a laser beam: the ion can loose $m$ vibrational quanta and make the transition from excited to ground or can gain $m$ vibrational quanta and make the transition from ground to excited; the ion can loose $m$ vibrational quanta and make the transition from ground to excited, or gain $m$ vibrational quanta and make the transition from excited to ground. The last possible transition depicted in the figure shows a transition from ground to excited (or vice versa) with no exchange of vibrational quanta.

Figure 17  Plot of the (atomic) inversion, $W(t) = P_2 - P_1$, the probability to find the ion in its excited state minus the probability to find it in the ground state, for an ion initially in its excited state and the vibrational state in a coherent state, with $\alpha = 5$.

Figure 18  Plot of the (atomic) inversion, $W(t) = P_2 - P_1$, the probability to find the ion in its excited state minus the probability to find it in the ground state, for an ion initially in its excited state and the vibrational state in a thermal distribution with $\bar{n} = 2$.

Figure 19  Possible situations we can have if we filter number states with the proposals of this section.

Figure 20  We plot $\Delta \phi_{\text{coh}}$ (solid line), and $1/2\rho^2$ (dash line) as a function of $\rho$.

Figure 21  We plot $1/(4\rho^2 + 3/\pi^2)$ (solid line), $\Delta \phi_{\text{coh}}^{P-B}$ (dash line) and $\Delta \phi_W$ (dot-dash line) as a function of $\rho$. 