Number phase quasiprobability distribution functions

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Over the years, quantum phase space distributions have not only been useful tools that allow to transcribe operator equations into c-number partial differential equations, but have proved to have many other uses, for instance in the reconstruction of the quantum state of a cavity field or the vibrational motion of ions. Among the most important ones, one can mention the Wigner distribution function, the Q function and the Glauber-Sudarshan P-function, which belong to a more general one-parameter family of quantum phase space distributions.

Here we would like to show that it can be constructed a Wigner distribution function for phase [1] and number in an analogous way in which is defined for the position-momentum variables.

We start by writing the Wigner function for position and momentum in the form

\[ W(\alpha) = \int C(\beta) \exp(\alpha^{\dagger} \beta - \alpha \beta^{\dagger}) d^{2} \beta, \]  

(1)

where the characteristic function, \( C(\beta) \) is defined as

\[ C(\beta) = \text{Tr}[\hat{D}(\beta)\hat{\rho}], \]

(2)

with \( \hat{\rho} \) the system's density matrix, \( \hat{D}(\beta) = \exp(\beta^{\dagger} \hat{a} - \beta^{*} \hat{a}^{\dagger}) \) the Glauber's displacement operator, and \( \hat{a} \) and \( \hat{a}^{\dagger} \) the annihilation and creation operators of the quantized field, respectively [2]. One can further write it in terms of the position and momentum operators such that the characteristic function times the Kernel reads

\[ \tilde{C}(\beta_x, \beta_p) = \text{Tr}[\hat{\rho}e^{-i\sqrt{2}(\beta_x \hat{p} - \beta_p \hat{x})} e^{\alpha^{\dagger} \beta - \alpha \beta^{\dagger}}], \]  

(3)

where we have set the frequency of the quantized electromagnetic field and \( \hbar \) equal to one.

In analogy to equation (3) we can define the function

\[ \tilde{C}_{\hat{n}}(k, \theta) = \frac{1}{2} \text{Tr}\left[ (\hat{D}_{\hat{n}}(k, \theta)e^{-i(k\phi - n\theta)} + c.c.) \hat{\rho} \right], \]

(4)

where

\[ \hat{D}_{\hat{n}}(k, \theta) = e^{i\theta k} e^{-i\hbar \hat{V}^{\dagger}} \]

(5)

with \( \hat{V}^{\dagger} = \sum_{k=0}^{\infty} |k + 1\rangle \langle k| \) the Susskind-Glogower operator. Because the Susskind-Glogower formalism fails in the phase description of the electromagnetic field with small photon numbers the unitarity of \( \hat{V} \) is spoiled. Also the fact that there is not a well defined phase operator, one can not use an expression of the form \( \exp[i(k \Phi - \phi \hat{n})] \), and we use instead a "factorized" form in Eq. (5).

Note that in order to produce a real Wigner function we added the complex conjugate in (4) (because \( n \) can not be a negative integer). Eq. (1) does not have this problem because the integrations over \( \beta_x \) and \( \beta_p \) are from \(-\infty \) to \( \infty \). By writing the density matrix in the number state basis,

\[ \hat{\rho} = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} Q_{m,l} |m\rangle \langle l|, \]

(6)

we obtain

\[ \tilde{C}_{\hat{n}}(k, \theta) = \frac{e^{i\theta k}}{2} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} Q_{m,l} \text{Tr}[\hat{V}^{\dagger} \hat{V}] e^{-i\hbar \hat{a}^{\dagger} |m\rangle \langle l|] \times e^{-i(k\phi - n\theta)} + c.c., \]

(7)

The double integration over the whole phase space in (1) becomes here a sum and a single integration

\[ W(n, \phi) = \frac{1}{(2\pi)^{2}} \sum_{k=-\infty}^{\infty} \int_{0}^{2\pi} \tilde{C}_{\hat{n}}(k, \theta) d\theta. \]

(8)

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Inserting equation (7) into (8) we obtain

\[ W(n, \phi) = \frac{1}{4\pi} \sum_{k=-n}^{\infty} \left( Q_{n,n+k} e^{-ik\phi} + Q_{n+k,n} e^{ik\phi} \right) . \tag{9} \]

It is easy to show that integrating (9) over the phase \( \phi \)

\[ \int_0^{2\pi} W(n, \phi) d\phi = Q_{n,n} = P(n), \tag{10} \]

gives the photon distribution. And adding (9) over \( n \)

\[ P(\phi) = \sum_{n=0}^{\infty} W(n, \phi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_{n,m} e^{-i(m-n)\phi} \tag{11} \]

produces the correct phase distribution. It is worth to note that for a number state \( |M\rangle \) equation (9) reduces to \( W(n, \phi) = \delta_{n,M}/2\pi \), i.e. it is different from zero only for \( n = M \) as it should be expected.

References