Nonclassicality of Quantum States

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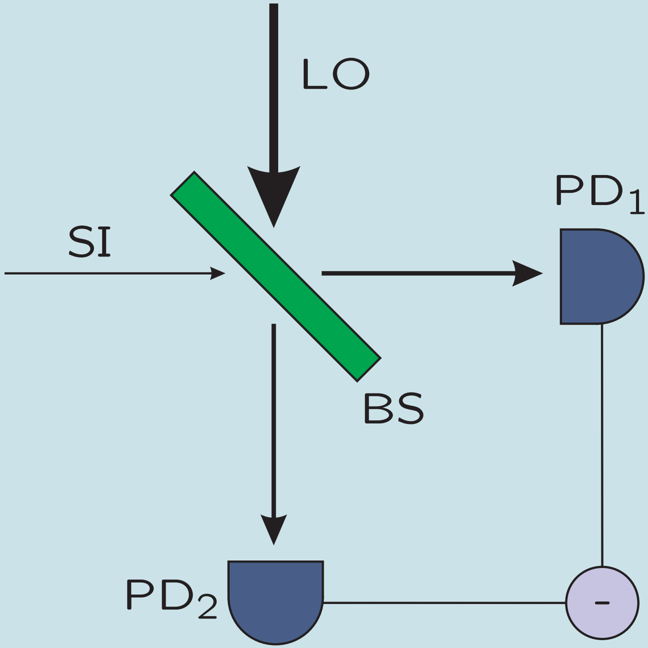
Characterization of quantum states

Balanced homodyne detection:



Introduction

Measured quantities:

- Difference statistics ⇔ quadrature operator:

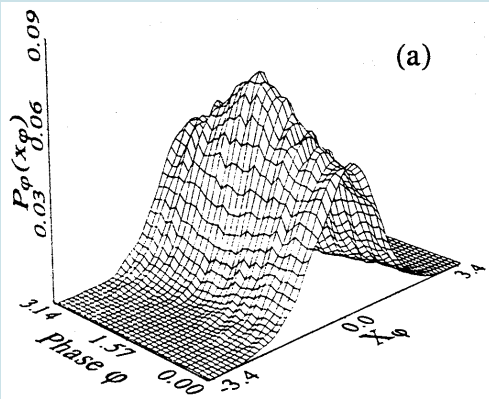
$$\hat{x}_\varphi = \hat{a} e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}$$

- Perfect detection, strong LO:

$$P_{\Delta m} = \frac{1}{|\alpha|} p\left(x = \frac{\Delta m}{|\alpha|}, \varphi\right)$$

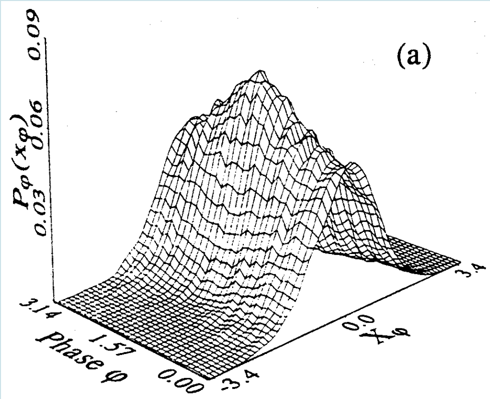
Introduction

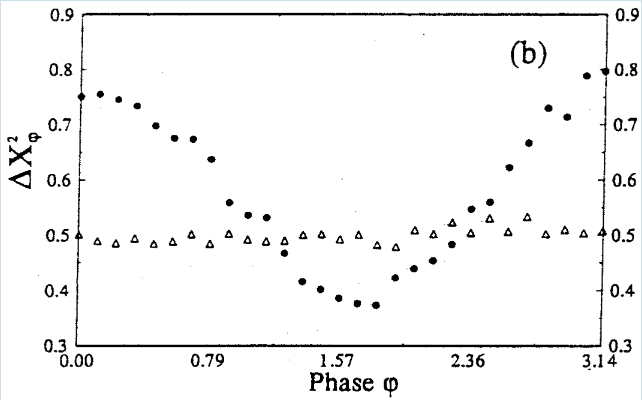
Experimental realization:



Introduction

Experimental realization:



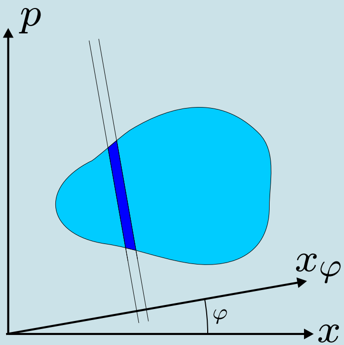


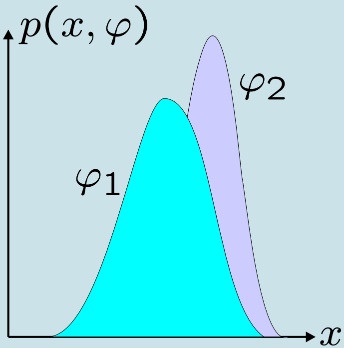
→ squeezed vacuum state

[Smithey, Beck, Raymer, Faridani, Phys. Rev. Lett. **70**, 1244 (1993)]

Introduction

Tomographic quantum-state reconstruction:





- measuring $p(x,\varphi)$ for $\varphi \ldots \varphi + \pi$

$\rightarrow$ Wigner function: $W(\alpha)$

$\rightarrow$ Density matrix

[K. Vogel and H. Risken, Phys. Rev. A**40**, 2847 (1989)]

Nonclassical phase-space functions

P-representation of the density operator:

$$\hat{\rho} = \int d^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha|$$

- expectation values:

$$\langle : \hat{F}(\hat{a}^\dagger, \hat{a}) : \rangle = \int d^2\alpha P(\alpha) F(\alpha^*, \alpha)$$

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Correspond to classical mean values:

(1) ”subtracting” ground-state noise via $\hat{F} \longrightarrow : \hat{F} :$

(2) P corresponds to classical probability: $P(\alpha) \equiv P_{\rm cl}(\alpha)$

[U.M. Titulaer and R.J. Glauber, Phys. Rev. **140**, B676 (1965)]

Nonclassical phase-space functions

A state is nonclassical, if:

(a) ground-state noise is substantial;
cf. also nonclassicality in weak measurements

[L.M. Johansen, Phys. Lett. A **329**, 184 (2004)]

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– The only classical pure states are coherent ones!

[M. Hillery, Phys. Lett. A**111**, 409 (1985)]

– Squeezing: $\langle : (\Delta\hat{x}_\varphi)^2 : \rangle < 0$

– sub-Poissonian photon statistics: $\langle : (\Delta\hat{n})^2 : \rangle < 0$

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- Squeezing: $\langle : (\Delta \hat{x}_\varphi)^2 : \rangle < 0$
- sub-Poissonian photon statistics: $\langle : (\Delta \hat{n})^2 : \rangle < 0$

- Sought: observable conditions for $P(\alpha) \neq P_{\mathrm{cl}}(\alpha)$
- Problem: $P(\alpha)$ may be strongly singular!

Nonclassical characteristic functions

Characteristic function of $P(\alpha)$:

$$\Phi(\beta) = \int d^2\alpha \, P(\alpha) \exp[(\alpha\beta^* - \alpha^*\beta)]$$

- Bochner Theorem (1933):

$\Phi(\beta)$ is a classical characteristic function, if and only if

$$\sum_{i,j=1}^{n} \Phi(\beta_i - \beta_j) \, \xi_i \, \xi_j^* \geq 0,$$

for any integer n and all complex β_i, ξ_k $(i, k = 1 \ldots n)$.

Nonclassical characteristic functions

- Define matrix: $\Phi_{ij} = \Phi(\beta_i - \beta_j)$
- Theorem:

  $\Phi(\beta)$ is a classical characteristic function, if and only if

$$D_k \equiv D_k(\beta_1, \ldots \beta_k) = \begin{vmatrix} 1 & \Phi_{12} & \cdots & \Phi_{1k} \\ \Phi^*_{12} & 1 & \cdots & \Phi_{2k} \\ \multicolumn{4}{c}{\dotfill} \\ \Phi^*_{1k} & \Phi^*_{2k} & \cdots & 1 \end{vmatrix} \geq 0$$

  for any order $k = 1, \ldots, +\infty$.

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  for any order $k = 1, \ldots, +\infty$.

$\Rightarrow$ $P(\alpha)$ is **not a probability** if and only if there exist values of k and β_k ($k = 2 \ldots \infty$) with

$$D_k(\beta_1, \ldots \beta_k) < 0$$

[Th. Richter and W. Vogel, Phys. Rev. Lett. **89**, 283601 (2002)]

Nonclassical characteristic functions

Observable characteristic functions of quadratures

$$G(k,\varphi) = G_{\mathrm{gr}}(k)\,\Phi(ike^{-i\varphi}),$$

with $\Phi = 1$ in the ground state

- First-order nonclassicality:

[W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000)]

$$|G(k,\varphi)| > G_{\mathrm{gr}}(k)$$

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$$|G(k,\varphi)| > G_{\mathrm{gr}}(k)$$

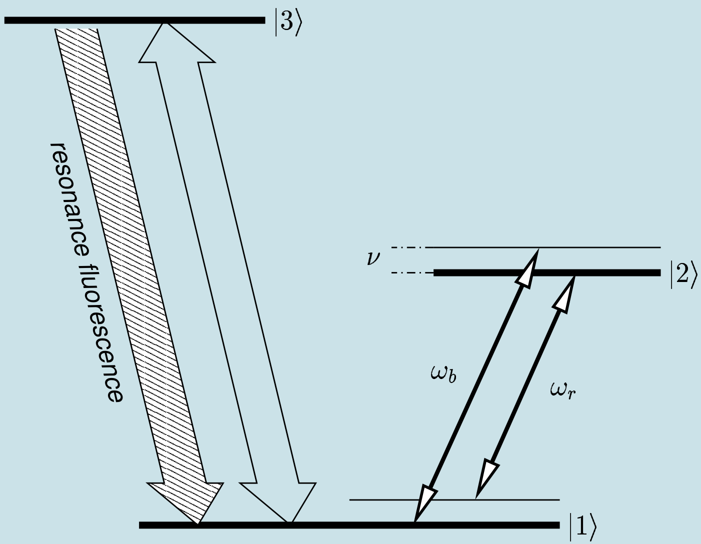
- applies to many nonclassical states:
  Fock, squeezed, even/odd coherent states, . . .
- experimental demonstration:
  mixture of a single photon with the vacuum state

$$\hat{\rho} = \eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0|$$

[A.I. Lvovsky and J.H. Shapiro, Phys. Rev. A **65**, 033830 (2002)]

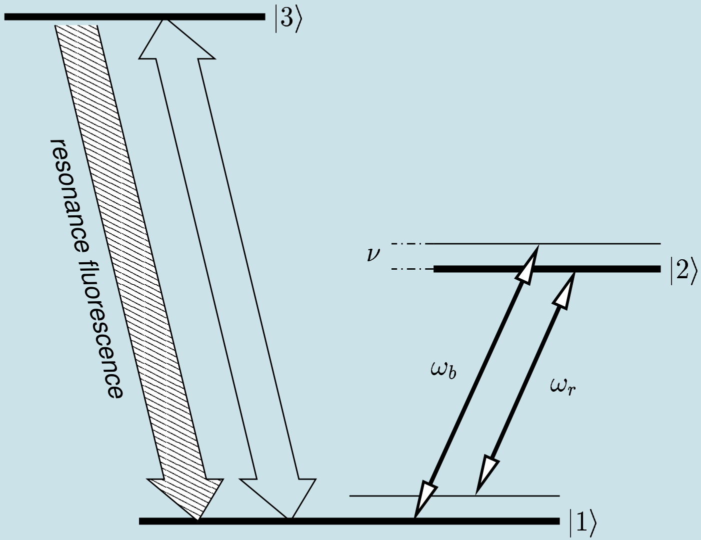
Nonclassical characteristic functions

Direct observation via fluorescence



Nonclassical characteristic functions

Direct observation via fluorescence



- Hamiltonian: $\hat{H}_{\mathrm{int}} = \frac{1}{2}\hbar\left(\Omega\hat{A}_{12} + \Omega^{*}\hat{A}_{21}\right)\hat{x}(\varphi)$

  [S. Wallentowitz and W. Vogel, Phys. Rev. Lett. **75**, 2932 (1995)]

$\Rightarrow$ experimental realization

[P.C. Haljan, K.-A. Brickman, L. Deslauriers, P.L. Lee, and C. Monroe (2004)]

General nonclassicality condition

Reformulation

- Hermitian Operator: $\hat{f}^\dagger \hat{f}$
- Normally ordered expectation value:

$$\langle : \hat{f}^\dagger \hat{f} : \rangle = \int d^2\alpha \, |f(\alpha)|^2 P(\alpha),$$

$\Rightarrow$ nonnegative for $P(\alpha) = P_{\text{cl}}(\alpha)$, for any operator $\hat{f}$

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- Quantum state nonclassical, iff there exists $\hat{f}$ with

$$\langle : \hat{f}^\dagger \hat{f} : \rangle < 0$$

$\Rightarrow$ various choices of representations of $\hat{f}$!

General nonclassicality condition

Sufficient Conditions for nonclassicality:

- Sub-Possonian number statistics:

$$\hat{f} \equiv \Delta\hat{n} = \hat{n} - \langle\hat{n}\rangle, \quad \hat{n} = \hat{a}^{\dagger}\hat{a}$$

$\Rightarrow$ condition:

$$\langle: \hat{f}^{\dagger}\hat{f} :\rangle \rightarrow \langle: (\Delta\hat{n})^2 :\rangle < 0$$

- Quadrature Squeezing:

$$\hat{f} \equiv \Delta\hat{x}_{\varphi} = \hat{x}_{\varphi} - \langle\hat{x}_{\varphi}\rangle, \quad \hat{x}_{\varphi} = \hat{a}e^{i\varphi} + \hat{a}^{\dagger}e^{-i\varphi}$$

$\Rightarrow$ condition:

$$\langle: (\Delta\hat{x}_{\varphi})^2 :\rangle < 0$$

General nonclassicality condition

Fourier representation

$$\hat{f} = \int d^2\alpha \, \underline{f}(\alpha) \, :\hat{D}(-\alpha):$$

- condition

$$\langle : \hat{f}^\dagger \hat{f} : \rangle < 0$$

- now reads as:

$$\int d^2\alpha \int d^2\beta \, \underline{f}(\alpha) \, \underline{f}^*(\beta) \, \Phi(\alpha - \beta) < 0$$

$\rightarrow$ continuous version of the Bochner condition!

$\rightarrow$ criteria for characteristic functions: special representation!

General nonclassicality condition

Taylor expansion

$$\hat{f} \equiv \hat{f}(\hat{A}, \hat{B}) = \sum_{n,m} f_{nm} : \hat{A}^n \hat{B}^m :$$

Choice of $\hat{A}$, $\hat{B}$ for complete description:

- Hermitian operators:

  (a) $\hat{A} = \hat{x}_\varphi$, $\hat{B} = \hat{p}_\varphi$, $\quad \hat{p}_\varphi \equiv \hat{x}_{\varphi+\pi/2}$

  (b) $\hat{A} = \hat{x}_\varphi$, $\hat{B} = \hat{n}$

- non-Hermitian operators:

  (c) $\hat{A} = \hat{a}^\dagger$, $\hat{B} = \hat{a}$

  $\Rightarrow$ different types of complete sets of criteria!

Nonclassical moments of two quadratures

Taylor expansion in quadratures

$$\hat{f} = f(\hat{x}_\varphi, \hat{p}_\varphi) = \sum_{n,m} f_{nm} \; : \hat{x}_\varphi^n \hat{p}_\varphi^m :$$

- nonclassicality condition

$$\langle : \hat{f}^\dagger \hat{f} : \rangle \Rightarrow \sum_{n,m,k,l} f_{nm} f_{kl}^* M_{nm,kl}(\varphi) < 0$$

where

$$M_{nm,kl}(\varphi) = \langle : \hat{x}_\varphi^{n+k} \hat{p}_\varphi^{m+l} : \rangle$$

[E. Shchukin, Th. Richter, and W. Vogel, to be published]

Nonclassical moments of two quadratures

In terms of determinants:

- determinants under study:

$$d_\varphi^{(N)} = \begin{vmatrix} 1 & \langle : \hat{x}_\varphi : \rangle & \langle : \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi : \rangle & \langle : \hat{p}_\varphi^2 : \rangle & \ldots \\ \langle : \hat{x}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^3 : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^2 : \rangle & \ldots \\ \langle : \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi : \rangle & \langle : \hat{p}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^2 : \rangle & \langle : \hat{p}_\varphi^3 : \rangle & \ldots \\ \langle : \hat{x}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi^3 : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^4 : \rangle & \langle : \hat{x}_\varphi^3 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi^2 : \rangle & \ldots \\ \langle : \hat{x}_\varphi \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi^3 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^3 : \rangle & \ldots \\ \langle : \hat{p}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^2 : \rangle & \langle : \hat{p}_\varphi^3 : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi \hat{p}_\varphi^3 : \rangle & \langle : \hat{p}_\varphi^4 : \rangle & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

- Necessary and sufficient nonclassicality conditions: there exist values of N ($N \geq 2$) and φ with

$$d_\varphi^{(N)} < 0$$

Nonclassical moments of two quadratures

Sufficient conditions:

(1) Restriction to second-order determinant:

$$d_\varphi^{(2)} = \langle : (\Delta \hat{x}_\varphi)^2 : \rangle < 0$$

$\longrightarrow$ quadrature squeezing!

Nonclassical moments of two quadratures

Sufficient conditions:

(1) Restriction to second-order determinant:

$$d_\varphi^{(2)} = \langle : (\Delta \hat{x}_\varphi)^2 : \rangle < 0$$

$\longrightarrow$ quadrature squeezing!

(2) Third-order determinant:

$$d_\varphi^{(3)} = \langle : (\Delta \hat{x}_\varphi)^2 : \rangle \langle : (\Delta \hat{p}_\varphi)^2 : \rangle - \langle : \Delta \hat{x}_\varphi \Delta \hat{p}_\varphi : \rangle^2 < 0$$

$\longrightarrow$ moments of two quadratures, but:

$$d_\varphi^{(3)} = \langle : (\Delta \hat{x}_\varphi)^2 : \rangle_{\min} \langle : (\Delta \hat{p}_\varphi)^2 : \rangle_{\max}$$

$\longrightarrow$ no new effect!

Nonclassical moments of two quadratures

(3) Elimination of one quadrature:

$$q_\varphi^{(n)} = \begin{vmatrix} 1 & \langle : \hat{x}_\varphi : \rangle & \dots & \langle : \hat{x}_\varphi^{n-1} : \rangle \\ \langle : \hat{x}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 : \rangle & \dots & \langle : \hat{x}_\varphi^{n} : \rangle \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \langle : \hat{x}_\varphi^{n-1} : \rangle & \langle : \hat{x}_\varphi^{n} : \rangle & \dots & \langle : \hat{x}_\varphi^{2n-2} : \rangle \end{vmatrix}$$

$\rightarrow$ nonclassicality conditions due to Agarwal:

$$q_\varphi^{(n)} < 0$$

[G.S. Agarwal, Opt. Commun. **95**, 109 (1993)]

Nonclassical moments of two quadratures

(4) Sub-determinants, for example:

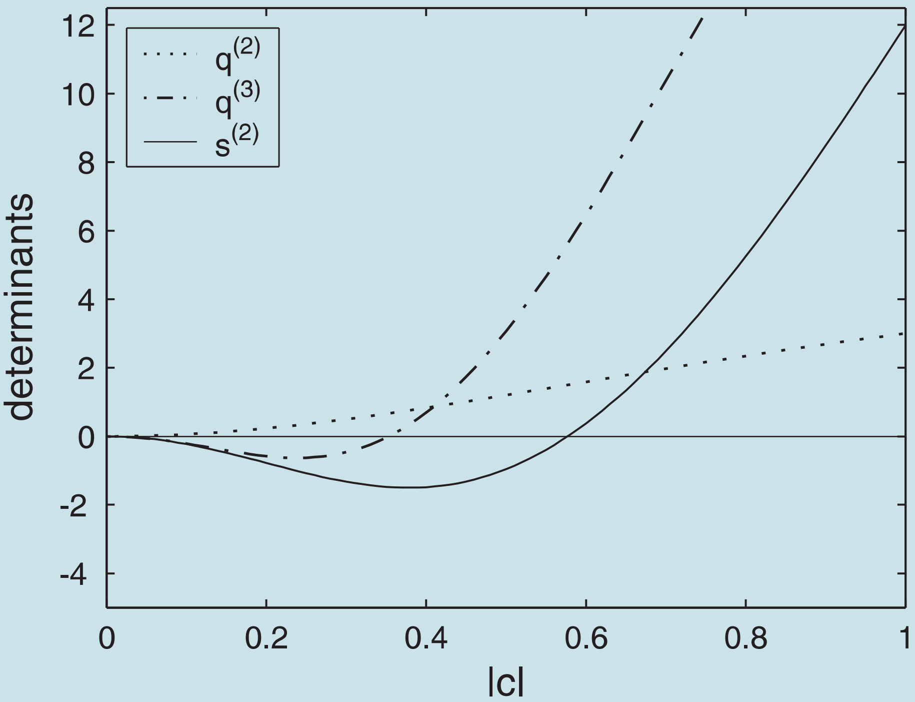
$$s_\varphi^{(2)} = \begin{vmatrix} \langle : \hat{x}_\varphi^2 : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle \\ \langle : \hat{x}_\varphi^2 \hat{p}_\varphi : \rangle & \langle : \hat{x}_\varphi^2 \hat{p}_\varphi^2 : \rangle \end{vmatrix} < 0$$

$\rightarrow$ Illustration for the quantum state:

$$|\psi\rangle = \frac{|0\rangle + c\,|3\rangle}{\sqrt{1 + |c|^2}}$$

$\rightarrow$ nonclassical for a larger parameter range!

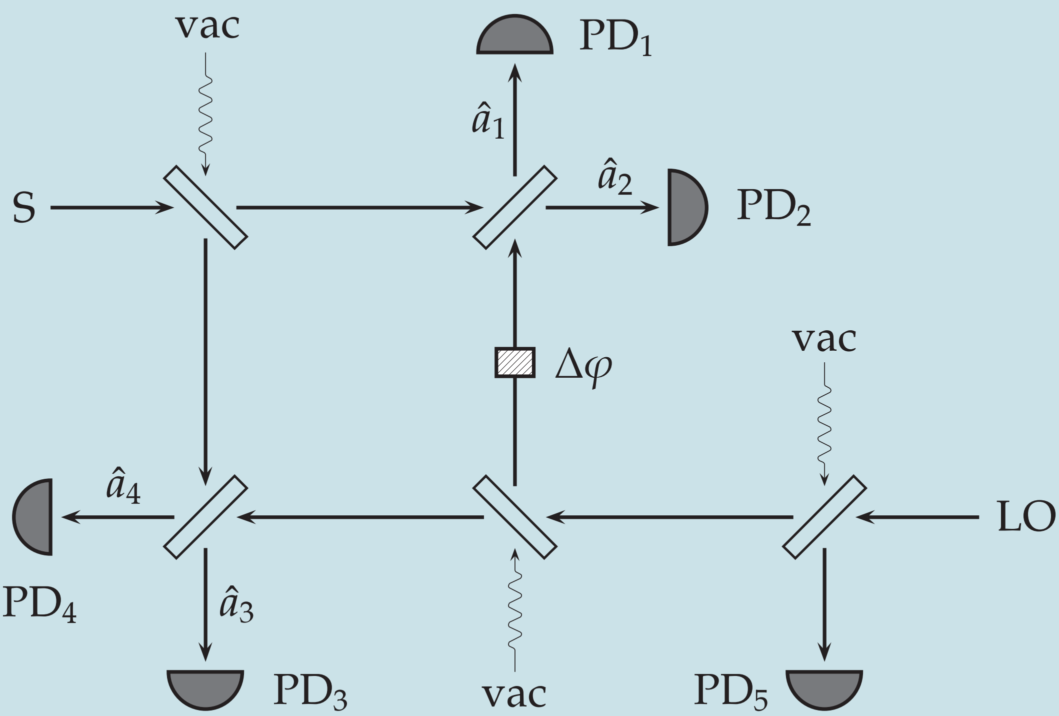
Nonclassical moments of two quadratures



→ no squeezing ($q^{(2)} > 0$), but $q^{(3)}$, $s^{(2)} < 0$!

Measuring moments of two quadratures

Basic measurement scheme:



see also [J.W. Noh, A. Fougéres, and L. Mandel, Phys. Rev. Lett. **67**, 1426 (1991)]

Measuring moments of two quadratures

- effective photon-number operators:

$$\hat{n}_{1,2} = \frac{1}{4}\left(\hat{n} \pm |\alpha|\,\hat{p}_\varphi + |\alpha|^2\right)$$
$$\hat{n}_{3,4} = \frac{1}{4}\left(\hat{n} \pm |\alpha|\,\hat{x}_\varphi + |\alpha|^2\right)$$

- detecting correlations, such as:

$\langle : \hat{n}_i \hat{n}_j : \rangle$, $\langle : \hat{n}_i \hat{n}_j \hat{n}_k : \rangle$, ...

- advantage: insensitive to efficiencies of detectors!
- extension to high orders possible!

[M. Beck, C. Dorrer, I. A. Walmsley, Phys. Rev. Lett. **87**, 253601 (2001)]

Nonclassical number-quadrature moments

Taylor expansion in number and quadrature

- Reformulate the condition $\langle : \hat{f}^\dagger \hat{f} : \rangle < 0$
- with the representation

$$\hat{f} = f(\hat{x}_\varphi, \hat{n}) = \sum_{k,l} f_{kl} \, : \hat{x}_\varphi^k \hat{n}^l :$$

- Conditions in terms of number-quadrature moments:

$$M_{k,l} = \langle : \hat{x}_\varphi^k \hat{n}^l : \rangle$$

$\Rightarrow$ Homodyne correlation measurements

[W. Vogel, Phys. Rev. Lett. **67**, 2450 (1991); Phys. Rev. A**51**, 4160 (1995); H.J. Carmichael, H.M. Castro-Beltran, G.T. Foster, L.A. Orozco, Phys. Rev. Lett. **85**, 1855 (2000)]

Nonclassical number-quadrature moments

⇒ Observables of dissimilar types:
$\hat{x}_\varphi$ continuous and $\hat{n}$ discrete and non-negative!

⇒ Two different types of nonclassicality conditions:

$$\begin{vmatrix} 1 & M_{0,1} & M_{1,0} & \cdots \\ M_{0,1} & M_{0,2} & M_{1,1} & \cdots \\ M_{1,0} & M_{1,1} & M_{2,1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} < 0$$

$$\begin{vmatrix} 2M_{0,1} - M_{2,0} & 2M_{0,2} - M_{2,1} & 2M_{1,1} - M_{3,0} & \cdots \\ 2M_{0,2} - M_{2,1} & 2M_{0,3} - M_{2,2} & 2M_{1,2} - M_{3,1} & \cdots \\ 2M_{1,1} - M_{3,0} & 2M_{1,2} - M_{3,1} & 2M_{2,1} - M_{4,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} < 0$$

Comment on entangled states

Criteria for continuous variable entanglement

- Conditions based on second-order moments

  [L.M. Duan, G. Giedke, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000); R. Simon, Phys. Rev. Lett. **84**, 2726 (2000)]

- Negative partial transposition of density matrix

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- Negative partial transposition of density matrix

$\Rightarrow$ General test via NPT condition:

$$\langle \hat{f}^\dagger \hat{f} \rangle_{\mathrm{PT}} < 0$$

$\Rightarrow$ Fourier representation (two modes):

$$\hat{f} = \int d^2\alpha_1 \, \underline{f}(\alpha_1, \alpha_2) : \hat{D}(-\alpha_1)\hat{D}(-\alpha_2) :$$

Comment on entangled states

Complete condition for negative PT

- Discrete version of $\langle \hat{f}^\dagger \hat{f} \rangle_{\mathrm{PT}} < 0$:

$$\sum_{i,j=1}^{n} e^{\alpha_i^* \alpha_j + \beta_i^* \beta_j} \, \Phi(\alpha_i - \alpha_j, \beta_j^* - \beta_i^*) \, \xi_i \, \xi_j^* < 0$$

Comment on entangled states

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$$\sum_{i,j=1}^{n} e^{\alpha_i^* \alpha_j + \beta_i^* \beta_j} \, \Phi(\alpha_i - \alpha_j, \beta_j^* - \beta_i^*) \, \xi_i \, \xi_j^* < 0$$

$\Rightarrow$ Conditions for determinants of characteristic functions

$\Rightarrow$ Observable conditions

$\Rightarrow$ Systematic check of NPT for non-Gaussian continuous quantum states!

$\Rightarrow$ Only sufficient criterion for entanglement!

Summary

- Nonclassical P-functions
- Nonclassical characteristic functions
- Nonclassical conditions for quadrature moments
- Measurement of quadrature moments
- Nonclassical number-quadrature moments
- Criteria for NPT of entangled states