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Preface

As the title implies, the present technical report entails two purposes. The first one is a brief systematic mathematical exposition of the $\mathscr{Z}$-transform including both theoretical concepts and operations. Also, several examples of various degrees of difficulty are given. Both direct and inverse computation of $\mathscr{Z}$-transforms is explained in sufficient detail to make clear how these calculations are realized. The second aim is the application of this transform to the solution of second order ordinary difference equations in a similar way as it is done using the Laplace transform to solve initial value problems specified by a second order ordinary differential equation. Here too, a general framework is established in the case of second order difference equations, and solutions to numerical examples corresponding to initial value problems are fully developed to enhance comprehension of how a solution sequence is found.

The exposition given here, as background material for different scientific or engineering fields, can be used by advanced undergraduate or graduate students, teachers, and researchers. Readers are required to be familiar with some topics in higher mathematics such as complex variables, low order linear ordinary differential equations, and the basic elements of the Laplace transform as used to solve initial value problems in ordinary differential equations. It is important to remark that discrete or digital sequences can be treated also from the viewpoint of generating functions or from the standpoint of recurrence relations. However, to avoid unnecessary repetitions along with presenting a compact work, I have used only the discrete transform approach. Nevertheless, the given bibliographical references span the whole spectrum of ways in dealing with discrete sequences.

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Contents

1 Definition and Elementary Properties

1.1 The $\mathcal{Z}$-transform and its inverse

The $\mathcal{Z}$-transform (also written as z-transform), known mainly from work on *sampled data systems* due to Hurewicz, Ragazzini, and Zadeh, is derived as a discrete version from the Laplace transform, briefly named the $\mathcal{L}$-transform. In the same way as the $\mathcal{L}$-transform is applied to the analysis and synthesis of *continuous linear systems*, the use of the $\mathcal{Z}$-transform appears frequently in the analysis and design of *discrete linear systems*. Before we continue, it is important to point out that an alternative name for the topic presented here is known as the *theory of generating functions* (De Moivre, Euler, and Laplace) and is common to use it in other mathematical disciplines such as probability theory and combinatorial analysis.

Let $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous real function. Then, a *sampled* or *discretized* mapping, $\widetilde{\varphi} : \{nT \mid n \in \mathbb{N}, T \in \mathbb{R}^+\}$ where T denotes the *sampling period*, can be associated in a natural way to φ by means of the following expression,

$$\widetilde{\varphi}(t) = \varphi(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_{n=0}^{\infty} \varphi(nT)\delta(t - nT). \tag{1.1}$$

In Eq. (1.1), δ represents Dirac's unitary impulse function. If the scale t/T is used instead of t, then the domain of $\widetilde{\varphi}$ equals the set of natural numbers $\mathbb{N}$. Therefore, we can assign a *real sequence* of numbers equivalent to $\widetilde{\varphi}$. If φ and ψ are two continuous functions defined on $[0, \infty)$ such that $\varphi(t) \neq \psi(t)$ for all t, it may happen that $\widetilde{\varphi}(t) = \widetilde{\psi}(t)$ for some values of t (e. g., take $\varphi(t) = 1$ and $\psi(t) = \cos(2\pi t/T)$); in such case, to distinguish between the sequences associated to $\widetilde{\varphi}$ and $\widetilde{\psi}$, the period T must be lowered to produce a finer sampling.

Applying the Laplace transform to the sampled function $\widetilde{\varphi}$ we have that,

$$\mathcal{L}\{\widetilde{\varphi}(t)\} = \widetilde{\Phi}(s) = \sum_{n=0}^{\infty} \varphi(nT)\mathcal{L}\{\delta(t - nT)\} = \sum_{n=0}^{\infty} \varphi(nT)e^{-nTs}. \tag{1.2}$$

Since, $\widetilde{\Phi}(s) = \widetilde{\Phi}(s + 2\pi ni)$ with $n \in \mathbb{N}$ and $i = \sqrt{-1}$, then $\widetilde{\Phi}$ is a periodic function of s. Making the change of variable, $s = T^{-1}\ln z$ or $z = e^{sT}$, we obtain the formal expression that defines the *$\mathcal{Z}$-unilateral transform*. In symbols,

$$\mathcal{Z}\{\varphi(n)\} = \widetilde{\Phi}(s) \mid_{s=T^{-1}\ln z} = \sum_{n=0}^{\infty} \varphi(nT) z^{-n}. \tag{1.3}$$

It is customary to use the following notation:

$$\mathcal{Z}\{\varphi\} = \mathcal{Z}\{\varphi(n)\} = \Phi(z) = \sum_{n=0}^{\infty} \varphi(n) z^{-n}, \tag{1.4}$$

where the period T has been eliminated since it is a fixed known value. The correspondence between φ and Φ under $\mathcal{Z}$ will be referred as a *$\mathcal{Z}$-transform pair* symbolized by $\varphi(n) \leftrightarrow \Phi(z)$. Notice how the $\mathcal{Z}$-transform is just a *complex power series* whose coefficients are the sampled values of the transformed function. Being s a complex number expressed by $s = \sigma + i\omega$, where σ and ω are real numbers, we see that $|z| = |e^{sT}| = |e^{\sigma T}||e^{i\omega T}| = |e^{\sigma T}|$. The geometrical meaning of this result can be explained as follows in reference to Fig. 1 which depicts the complex mapping $z = e^{sT}$ from $\mathbb{C}$ to itself. If $\sigma < 0$ then $|z| < 1$, the left complex half-plane corresponds to the interior of the complex unit circle (centered at the origin). Similarly, if $\sigma > 0$ then $|z| > 1$, the right complex half-plane goes to the exterior of the complex unit circle. In the case that $\sigma = 0$ then $|z| = 1$, thus the imaginary axis $i\omega$ maps to the circumference of the complex unit circle.

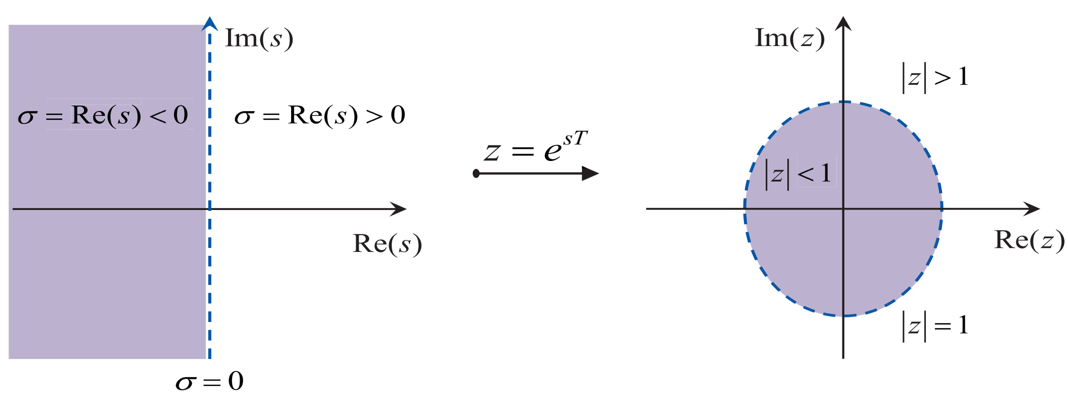

Figure 1: Corresponding convergence regions in the $\mathbb{C}$-plane of the $\mathcal{L}$ and the $\mathcal{Z}$-transforms.

In other words, Fig. 1 shows the relation between *convergence regions* specified, respectively, by the $\mathcal{L}$ and $\mathcal{Z}$ transforms. The notion of *stability*, as used in control theory, in a discrete linear system is closely related to the convergence radius of the power series $\Phi(z)$ given by Eq. (1.4). Associated to the "direct" $\mathcal{Z}$-transform, the *inverse $\mathcal{Z}$-transform* will always give a discrete function $\varphi(n)$ for $n \geq 0$. Its formula is given by the complex contour integral,

$$\varphi(n) = \mathcal{Z}^{-1}\{\Phi(z)\} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz, \tag{1.5}$$

where $\mathcal{C}$ is a circular path in the complex plane enclosing all singularities of $\Phi(z)z^{n-1}$, equivalently, all points such that $\Phi(z)z^{n-1} \to \infty$. The equality given by Eq. (1.5) can be obtained in the following way. The Laplace transform inversion formula establishes that,

$$\varphi(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) e^{st} ds. \tag{1.6}$$

Then setting $t = n$ and changing e^s by z (i.e., $s = \ln z$) we have

$$\varphi(n) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) e^{sn} ds = \frac{1}{2\pi i}\int_{e^{\sigma-i\infty}}^{e^{\sigma+i\infty}} \Phi(\ln z) z^n d(\ln z), \tag{1.7}$$

from which Eq. (1.5) follows.

1.2 Elementary properties

Next, we verify various important properties of the $\mathcal{Z}$-transform.

Linearity. Assume given a finite sequence of transform pairs, $c_k\varphi_k(n) \leftrightarrow c_k\Phi_k(z)$ where $k = 1, \ldots, m$ and $c_k \in \mathbb{R}$ for all k. Hence,

$$\mathcal{Z}\Big\{\sum_{k=1}^{m} c_k\varphi_k(n)\Big\} = \sum_{k=1}^{m} \mathcal{Z}\{c_k\varphi_k(n)\} = \sum_{k=1}^{m} c_k\mathcal{Z}\{\varphi_k(n)\} = c_k\sum_{k=1}^{m} \Phi_k(z). \tag{1.8}$$

For $m = 2$, we see that,

$$\begin{aligned}\mathcal{Z}\{c_1\varphi_1(n) + c_2\varphi_2(n)\} &= \sum_{n=0}^{\infty}[c_1\varphi_1(n) + c_2\varphi_2(n)]z^{-n} \\ &= c_1\sum_{n=0}^{\infty}\varphi_i(n)z^{-n} + c_2\sum_{n=0}^{\infty}\varphi_2(n)z^{-n} = c_1\Phi_1(z) + c_2\Phi_2(z).\end{aligned}$$

Notice that linearity may be extended to the case in which the scalars c_k are complex numbers. Thus, if we consider that $c_k = a_k + ib_k$ where $a_k, b_k \in \mathbb{R}$ for each k, then

$$\mathcal{Z}\Big\{\sum_{k=1}^{m}(a_k + ib_k)\varphi_k(n)\Big\} = \mathcal{Z}\Big\{\sum_{k=1}^{m} a_k\varphi_k(n)\Big\} + i\mathcal{Z}\Big\{\sum_{k=1}^{m} b_k\varphi_k(n)\Big\}. \tag{1.9}$$

From Eq. (1.9) we conclude easily that the $\mathcal{Z}$-transform of the real or imaginary part of a complex sequence gives the same result if we take, respectively, the real or imaginary part of the $\mathcal{Z}$-transform of the complex sequence. In other words, the $\mathcal{Z}$ and **Re**, or the $\mathcal{Z}$ and **Im** operators commute.

Multiplication by n. Assume as known the transform pair $\varphi(n) \leftrightarrow F(z)$. If $n > 0$ then,

$$\begin{aligned}\mathcal{Z}\{n\varphi(n)\} &= \sum_{n=0}^{\infty} n\varphi(n)z^{-n} = z\sum_{n=0}^{\infty}\varphi(n)nz^{-n-1} = z\sum_{n=0}^{\infty}\varphi(n)\Big(-\frac{dz^{-n}}{dz}\Big)\\ &= -z\frac{d}{dz}\sum_{n=0}^{\infty}\varphi(n)z^{-n} = -z\frac{dF(z)}{dz},\end{aligned}$$

or, equivalently, this result can be written as the transform pair

$$\boxed{n\varphi(n) \leftrightarrow -zF'(z)} \tag{1.10}$$

Note that the left member function φ can be considered multiplied by the identity function on $\mathbb{N}$, where $\mathrm{id}(n) = n$.

Division by $n + a$. We determine the $\mathcal{Z}$-transform of the function $\varphi(n)$ multiplied by the function $1/(n + a)$ where $a \in \mathbb{R}$ and $a \neq -n$. First, we observe that

$$\int_0^z \zeta^{-n-a-1}d\zeta = \frac{\zeta^{-n-a}}{-n-a}\Big|_0^z = \frac{z^{-n}z^{-a}}{-(n+a)},$$

therefore,

$$\begin{aligned}\mathcal{Z}\Big\{\frac{\varphi(n)}{n+a}\Big\} &= \sum_{n=0}^{\infty}\frac{\varphi(n)}{n+a}z^{-n} = \sum_{n=0}^{\infty}\varphi(n)z^{a}\Big(-\int_0^z \zeta^{-n-a-1}d\zeta\Big)\\ &= -z^{a}\int_0^z \zeta^{-a-1}\Big(\sum_{n=0}^{\infty}\varphi(n)\zeta^{-n}\Big)d\zeta.\end{aligned}$$

Hence the corresponding transform pair is given by,

$$\boxed{\frac{\varphi(n)}{n+a} \leftrightarrow -z^a \int_0^z \frac{\Phi(\zeta)}{\zeta^{a+1}} d\zeta} \tag{1.11}$$

Multiplication by a^n. Again, suppose that the transform pair $\varphi(n) \leftrightarrow F(z)$ is known. Thus, if $n > 0$ and $a > 0$ then,

$$\mathcal{Z}\{a^n \varphi(n)\} = \sum_{n=0}^{\infty} a^n \varphi(n) z^{-n} = \sum_{n=0}^{\infty} \varphi(n) \left(\frac{z}{a}\right)^{-n}.$$

Therefore,

$$\boxed{a^n \varphi(n) \leftrightarrow F\left(\frac{z}{a}\right)} \tag{1.12}$$

The last three basic transform pairs show the interplay between algebraic operations in the sequence domain and analytic or scale change operations in the transform domain. For the remaining set of properties of the $\mathcal{Z}$ operator we will take into consideration the following elementary sequences. The *unitary impulse* or *Dirac sequence* and the *unitary step* or *Heaviside sequence* are defined, respectively by,

$$\delta(n) = \begin{cases} 1 \Leftrightarrow n = 0 \\ 0 \Leftrightarrow n \neq 0 \end{cases} \quad \text{and} \quad u(n) = \begin{cases} 1 \Leftrightarrow n \geq 0 \\ 0 \Leftrightarrow n < 0 \end{cases}. \tag{1.13}$$

In similar fashion, the unitary impulse and the unitary step sequences *translated* (displaced) at m are specified as

$$\delta(n-m) = \begin{cases} 1 \Leftrightarrow n = m \\ 0 \Leftrightarrow n \neq m \end{cases} \quad \text{and} \quad u(n-m) = \begin{cases} 1 \Leftrightarrow n \geq m \\ 0 \Leftrightarrow n < m \end{cases}. \tag{1.14}$$

It is common to extend the domain of both of these elementary sequences to the set of all integers. Thus, $\mathrm{Dom}(\delta) = \mathrm{Dom}(u) = \mathbb{Z}$; also $m \in \mathbb{Z}$, and by definition, δ and u are *binary sequences* since $\mathrm{Im}(\delta) = \mathrm{Im}(u) = \{0, 1\}$. It is not difficult to verify the equality $\delta(n) = u(n) - u(n-1)$ that relates the impulse and step sequences. Since,

$$\sum_{m \neq n} \varphi(m)\delta(n-m) + \varphi(n)\delta(n-n) = 0 + \varphi(n) \cdot 1 = \varphi(n),$$

we can represent any sequence or discrete function φ as an *infinite linear combination* of displaced unitary impulses weighted by each of the function samples, i.e., for any $n \geq 0$,

$$\varphi(n) = \sum_{m=-\infty}^{\infty} \varphi(m)\delta(n-m) = \sum_{m\in\mathbb{Z}} \varphi(m)\delta(n-m) \tag{1.15}$$

We end this section with two additional basic properties used to specify backward or forward translation (displacement) by several units. Accordingly we have:

Backward Shift or Delay. Assume that $\varphi(n) = 0$ if $n < 0$ and that $m > 0$. Then,

$$\begin{aligned}\mathcal{Z}\{\varphi(n-m)\} &= \mathcal{Z}\{\varphi(n-m)u(n-m)\} = \sum_{n=0}^{\infty} \varphi(n-m)u(n-m)z^{-n} \\ &= \sum_{n=m}^{\infty} \varphi(n-m)z^{-n} = \sum_{k=0}^{\infty} \varphi(k)z^{-m-k} = z^{-m}\sum_{k=0}^{\infty} \varphi(k)z^{-k},\end{aligned}$$

from which,

$$\varphi(n-m) \leftrightarrow z^{-m}F(z) \tag{1.16}$$

Forward Shift orAdvance. Suppose that $m > 0$, then $-m < 0$; thus, if $n \geq -m$ we obtain by definition that $u(n+m) = u(n-(-m)) = 1$. Therefore,

$$\begin{aligned}\mathcal{Z}\{\varphi(n+m)u(n+m)\} &= \sum_{n=0}^{\infty} \varphi(n+m)u(n+m)z^{-n} = \sum_{n=0}^{\infty} \varphi(n+m)z^{-n} \\ &= \sum_{k=m}^{\infty} \varphi(k)z^{-k+m} = z^{m}\sum_{k=m}^{\infty} \varphi(k)z^{-k},\end{aligned}$$

after changing $n+m$ by k. Also, we can see that,

$$\sum_{k=0}^{\infty} \varphi(k)z^{-k} = \sum_{k=0}^{m-1} \varphi(k)z^{-k} + \sum_{k=m}^{\infty} \varphi(k)z^{-k}.$$

Therefore, we obtain the transform pair shown next and developed formulas for $m = 1$ to $m = 3$ are provided below.

$$\varphi(n+m) \leftrightarrow z^{m}F(z) - \sum_{k=0}^{m-1} \varphi(k)z^{m-k} \tag{1.17}$$

$$\varphi(n+1) \leftrightarrow z[F(z) - \varphi(0)],$$
$$\varphi(n+2) \leftrightarrow z^2[F(z) - \varphi(0)] - z\varphi(1),$$
$$\varphi(n+3) \leftrightarrow z^3[F(z) - \varphi(0)] - z^2\varphi(1) - z\varphi(2).$$

2 Transforms of Basic Sequences

In this section, we calculate in detail the $\mathcal{Z}$-transforms of the elementary unitary and step sequences as well as some other basic sequences such as exponentials and trigonometric sequences and products of them.

UNITARY IMPULSE sequence and shifted version. We see that,

$$\mathcal{Z}\{\delta(n)\} = \Delta(z) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = \sum_{n \neq 0} \delta(n) z^{-n} + \delta(0) z^{-0} = 1.$$

Hence, applying Eq. (1.16) we obtain the transform pair valid for $|z| > 0$,

$$\boxed{\delta(n-m) \leftrightarrow z^{-m}\Delta(z) = z^{-m}} \tag{2.1}$$

UNITARY STEP sequence and shifted version. Here, for $|z| > 1$ we have that,

$$\mathcal{Z}\{u(n)\} = \mathsf{U}(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}\,.$$

Again, using Eq. (1.16) we obtain the transform pair valid for $|z| > 1$ and $z \neq 0$,

$$\boxed{u(n-m) \leftrightarrow z^{-m}\mathsf{U}(z) = \frac{z^{1-m}}{z-1}} \tag{2.2}$$

We observe that the $\mathcal{Z}$-transform of the unitary impulse sequence is constant and its displaced version is well defined except at the origin of the complex plane. On the other hand, the $\mathcal{Z}$-transform of the unit step sequence is a rational function of z defined everywhere except on the unit circle. However its displaced version is defined everywhere except on the punctured unit circle at the origin if $m \neq 0$.

RAMP sequence. We make use of Eq. (1.10), so that for $|z| > 1$, we get

$$\mathcal{Z}\{nu(n)\} = -z\mathsf{U}'(z) = -z\frac{d}{dz}\Big(\frac{z}{z-1}\Big) = -z\Big[\frac{(z-1) - z}{(z-1)^2}\Big].$$

$$\Rightarrow \quad nu(n) \leftrightarrow \frac{z}{(z-1)^2} \tag{2.3}$$

PARABOLIC sequence. Using the previous result together with Eq. (1.10), we have that (for $|z| > 1$),

$$\mathcal{Z}\{n^2u(n)\} = -z\frac{d}{dz}\Big[\frac{z}{(z-1)^2}\Big] = -z\Big[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\Big].$$

$$\Rightarrow \quad n^2u(n) \leftrightarrow \frac{z(z+1)}{(z-1)^3} \tag{2.4}$$

n-th POWER sequence. Let $a \in \mathbb{R}^+$ be a positive constant, i.e., $a > 0$. Then, applying Eq. (1.12), we obtain the following result valid for $|z| > a$,

$$\mathcal{Z}\{a^nu(n)\} = \mathsf{U}\Big(\frac{z}{a}\Big) = \frac{za^{-1}}{za^{-1}-1}.$$

$$\Rightarrow \quad a^nu(n) \leftrightarrow \frac{z}{z-a} \tag{2.5}$$

EXPONENTIAL sequence. If we take $a = e^b$ for any $b \in \mathbb{R}$ not zero, then $a^n = e^{nb}$. Therefore, using the previous result displayed in Eq. (2.5), we get for $|z| > e^b$,

$$\Rightarrow \quad e^{nb}u(n) \leftrightarrow \frac{z}{z-e^b} \tag{2.6}$$

If in Eq. (2.6) we let $b = i\omega$ (an imaginary number) the corresponding transform pair for $|z| > 1$ becomes,

$$e^{in\omega}u(n) \leftrightarrow \frac{z}{z-e^{i\omega}}. \tag{2.7}$$

Furthermore, recalling Euler's identity $e^{i\omega} = \cos\omega + i\sin\omega$, then $e^{in\omega} = \cos(n\omega) + i\sin(n\omega)$. Hence, upon substitution, the right hand expression of Eq. (2.7) can be also written as

$$\begin{aligned}\frac{z}{z-e^{i\omega}} &= \frac{z}{z-\cos\omega - i\sin\omega}\left(\frac{z-\cos\omega+i\sin\omega}{z-\cos\omega+i\sin\omega}\right) = \frac{z(z-\cos\omega)+iz\sin\omega}{(z-\cos\omega)^2+\sin^2\omega}\\ &= \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1} + i\frac{z\sin\omega}{z^2-2z\cos\omega+1},\end{aligned}$$

from which taking the real and imaginary parts, we obtain respectively the following couple of transform pairs:

COSINE sequence. The corresponding transform pair follows immediately from the fact that, $\mathcal{Z}\{\mathbf{Re}(e^{in\omega}u(n))\} = \mathbf{Re}(\mathcal{Z}\{e^{in\omega}u(n)\})$.

$$\boxed{\cos(n\omega)u(n) \leftrightarrow \mathsf{C}(z) = \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}} \tag{2.8}$$

SINE sequence. Since, $\mathcal{Z}\{\mathbf{Im}(e^{in\omega}u(n))\} = \mathbf{Im}(\mathcal{Z}\{e^{in\omega}u(n)\})$, then

$$\boxed{\sin(n\omega)u(n) \leftrightarrow \mathsf{S}(z) = \frac{z\sin\omega}{z^2-2z\cos\omega+1}} \tag{2.9}$$

Additional transform pairs involving exponential and trigonometric sequences are now easily deduced from the given results. In Eq. (1.12) let $a = e^{-b}$ for any $b \in \mathbb{R}$ not equal to zero. Then, since $(e^{-b})^n\varphi(n)$ is transformed into $F(z/e^{-b})$ we get for the product of a sequence φ with a negative exponential, the pair,

$$\boxed{e^{-nb}\varphi(n) \leftrightarrow F(ze^b)} \tag{2.10}$$

Using this last result we compute explicitly the following transform pairs for products of a negative exponential and the trigonometric functions of cosine and sine. Specifically we have,

$$\begin{aligned}\mathcal{Z}\{e^{-nb}\cos(n\omega)u(n)\} = \mathsf{C}(ze^b) &= \frac{ze^b(ze^b-\cos\omega)}{(ze^b)^2-2ze^b\cos\omega+1}\\ &= \frac{ze^{2b}(z-e^{-b}\cos\omega)}{e^{2b}(z^2-2ze^{-b}\cos\omega+e^{-2b})} = \frac{z(z-e^{-b}\cos\omega)}{z^2-2ze^{-b}\cos\omega+e^{-2b}}\,.\end{aligned}$$

Analogously,

$$\begin{aligned}\mathcal{Z}\{e^{-nb}\sin(n\omega)u(n)\} = \mathsf{S}(ze^b) &= \frac{ze^b\sin\omega}{(ze^b)^2-2ze^b\cos\omega+1}\\ &= \frac{ze^{2b}e^{-b}\sin\omega}{e^{2b}(z^2-2ze^{-b}\cos\omega+e^{-2b})} = \frac{ze^b\sin\omega}{z^2-2ze^{-b}\cos\omega+e^{-2b}}\,.\end{aligned}$$

To find the transform pairs corresponding to the hyperbolic sequences we can apply the property of linearity established in Section 1.

Hyperbolic Cosine sequence. Computing the $\mathcal{Z}$-transform we obtain,

$$\begin{aligned}\mathcal{Z}\{\cosh(n\omega)u(n)\} &= \mathcal{Z}\left\{\frac{1}{2}(e^{n\omega}+e^{-n\omega})u(n)\right\} = \frac{1}{2}\Big[\mathcal{Z}\{e^{n\omega}u(n)\}+\mathcal{Z}\{e^{-n\omega}u(n)\}\Big]\\ &= \frac{1}{2}\left[\frac{z}{z-e^{\omega}}+\frac{z}{z-e^{-\omega}}\right] = \frac{1}{2}\left[\frac{z(z-e^{-\omega})+z(z-e^{\omega})}{z^2-ze^{\omega}-ze^{-\omega}+1}\right]\\ &= \frac{1}{2}\left[\frac{2z^2-z(e^{-\omega}+e^{\omega})}{z^2-z(e^{\omega}+e^{-\omega})+1}\right] = \frac{z^2-z\cosh\omega}{z^2-2z\cosh\omega+1},\end{aligned}$$

whence it follows that,

$$\cosh(n\omega)u(n) \leftrightarrow \mathsf{Ch}(z) = \frac{z(z-\cosh\omega)}{z^2-2z\cosh\omega+1} \tag{2.11}$$

Hyperbolic Sine sequence. Again, calculating the $\mathcal{Z}$-transform we get,

$$\begin{aligned}\mathcal{Z}\{\sinh(n\omega)u(n)\} &= \mathcal{Z}\left\{\frac{1}{2}(e^{n\omega}-e^{-n\omega})u(n)\right\} = \frac{1}{2}\Big[\mathcal{Z}\{e^{n\omega}u(n)\}-\mathcal{Z}\{e^{-n\omega}u(n)\}\Big]\\ &= \frac{1}{2}\left[\frac{z}{z-e^{\omega}}-\frac{z}{z-e^{-\omega}}\right] = \frac{1}{2}\left[\frac{z(z-e^{-\omega})-z(z-e^{\omega})}{z^2-ze^{\omega}-ze^{-\omega}+1}\right]\\ &= \frac{1}{2}\left[\frac{z(e^{\omega}-e^{-\omega})}{z^2-z(e^{\omega}+e^{-\omega})+1}\right] = \frac{z\sinh\omega}{z^2-2z\cosh\omega+1},\end{aligned}$$

therefore,

$$\sinh(n\omega)u(n) \leftrightarrow \mathsf{Sh}(z) = \frac{z\sinh\omega}{z^2-2z\cosh\omega+1} \tag{2.12}$$

We close this section with a formula for the $\mathcal{Z}$-transform of the sequence defined by $na^{n-1}u(n)$. The details are shown next.

$$\begin{aligned}\mathcal{Z}\{na^{n-1}u(n)\} &= \frac{1}{a}\mathcal{Z}\{n[a^n u(n)]\} = \frac{1}{a}\left(-z\frac{d}{dz}[a^n u(n)]\right)\\ &= -\frac{z}{a}\frac{d}{dz}\left(\frac{z}{z-a}\right) = -\frac{z}{a}\left[\frac{(z-a)-z}{(z-a)^2}\right].\end{aligned}$$

The first two equalities follow from the linearity and the multiplication by n properties given in Eqs. (1.8) and (1.10) respectively. The second set of equalities follow by applying the transform of an n-th power sequence as shown in Eq. (2.5) and by derivation with respect to z. The final transform pair is obtained by algebraic simplification. Hence, for $|z| > a$,

$$na^{n-1}u(n) \leftrightarrow \frac{z}{(z-a)^2} \tag{2.13}$$

3 Additional Properties

In this section the important theorems of the initial and final values are given as well as the $\mathcal{Z}$-transforms of other sequences such as those generated by a discrete sum of values and reciprocals of factorials.

Theorem 3.1 Initial Value. *Given the transform pair $\varphi(n) \leftrightarrow \Phi(z)$, then*

$$\lim_{n\to 0} \varphi(n) = \lim_{z\to\infty} \Phi(z). \tag{3.1}$$

Proof:

$$\lim_{n\to 0} \varphi(n) = \varphi(0) = \varphi(0) + \sum_{n=1}^{\infty} \lim_{z\to\infty} \left[\frac{\varphi(n)}{z^n}\right] = \varphi(0) + \lim_{z\to\infty} \sum_{n=1}^{\infty} \frac{\varphi(n)}{z^n}$$
$$= \lim_{z\to\infty} \left(\varphi(0) + \sum_{n=1}^{\infty} \varphi(n) z^{-n}\right) = \lim_{z\to\infty} \sum_{n=0}^{\infty} \varphi(n) z^{-n} = \lim_{z\to\infty} \Phi(z). \ \square$$

Theorem 3.2 Final Value. *Given the transform pair $\varphi(n) \leftrightarrow \Phi(z)$, then*

$$\lim_{n\to\infty} \varphi(n) = \lim_{z\to 1} \left(\frac{z-1}{z}\right) \Phi(z). \tag{3.2}$$

Proof:

$$\lim_{n\to\infty} \varphi(n) = \varphi(\infty) = \sum_{n=0}^{\infty} \varphi(n) - \sum_{n=1}^{\infty} \varphi(n-1) = \lim_{z\to 1} \left[\sum_{n=0}^{\infty} \varphi(n) z^{-n} - \sum_{n=1}^{\infty} \varphi(n-1) z^{-n}\right]$$
$$= \lim_{z\to 1} [\Phi(z) - z^{-1}\Phi(z)] = \lim_{z\to 1} (1 - z^{-1}) \Phi(z). \ \square$$

Finite Discrete Sum sequence. Let ψ be a sequence defined as the finite sum of values of another sequence given by φ, i.e.,

$$\psi(n) = \sum_{k=0}^{n} \varphi(k) = \varphi(0) + \varphi(1) + \cdots + \varphi(n-1) + \varphi(n).$$

Next we determine the corresponding $\mathcal{Z}$-transform, that is,

$$\mathcal{Z}\{\psi(n)\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \varphi(k)\right] z^{-n} = \left(\sum_{n=0}^{\infty} \varphi(n) z^{-n}\right) \left(\sum_{n=0}^{\infty} z^{-n}\right)$$
$$= \Phi(z) \mathsf{U}(z) = \left(\frac{z}{z-1}\right) \Phi(z) = \Psi(z).$$

The expression in the second double summation equality follows from Cauchy's product rule for power series written as,

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)x^n = \Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big),$$

expression in which we made the substitutions $a_k = \varphi(k)$, $b_{n-k} = 1$, and $x = z^{-1}$. Consequently, the corresponding transform pair is

$$\sum_{k=0}^{n} \varphi(k) \leftrightarrow \Big(\frac{z}{z-1}\Big)\Phi(z) \tag{3.3}$$

Reciprocal of the Factorial $(1/n!)$ sequence. In this case, $\mathcal{Z}\{1/n!\} = \sum_{n=0}^{\infty} z^{-n}/n! = \sum_{n=0}^{\infty}(z^{-1})^n/n!$; recalling that $e^x = \sum_{n=0}^{\infty} x^n/n!$ and letting $x = z^{-1}$, we obtain the transform pair,

$$\frac{u(n)}{n!} \leftrightarrow e^{1/z} \tag{3.4}$$

Reciprocal of the Double Factorial $[1/(2n)!]$ sequence. In this case, $\mathcal{Z}\{1/(2n)!\} = \sum_{n=0}^{\infty} z^{-n}/(2n)!$, Let $m = 2n$ then $n = m/2$; also, denote m even as $m \equiv 0$ and m odd as $m \equiv 1$, i.e., the short version of the congruence modulus 2 relationships given by $m \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$, respectively. In the following chain of equalities we first make the change of variable and add a zero using odd sums, next we complete the first sum over all $\mathbb{N}$ and add a zero again using even sums to match the algebraic form of the second sum, and finally we complete it once more over all $\mathbb{N}$. In the last step, observe that $(-1)^m = 1$ for any $m \equiv 0$. Thus, we have,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^{-n}}{(2n)!} &= \sum_{m\equiv 0}^{\infty}\frac{z^{-m/2}}{m!} + \sum_{m\equiv 1}^{\infty}\frac{z^{-m/2}}{m!} - \sum_{m\equiv 1}^{\infty}\frac{z^{-m/2}}{m!}\\ &= \sum_{m\equiv 0}^{\infty}\frac{z^{-m/2}}{m!} + \sum_{m\equiv 1}^{\infty}\frac{(-1)^m z^{-m/2}}{m!} + \sum_{m\equiv 0}^{\infty}\frac{(-1)^m z^{-m/2}}{m!} - \sum_{m\equiv 0}^{\infty}\frac{(-1)^m z^{-m/2}}{m!}\\ &= \sum_{m=0}^{\infty}\frac{z^{-m/2}}{m!} + \sum_{m=0}^{\infty}\frac{(-1)^m z^{-m/2}}{m!} - \sum_{m\equiv 0}^{\infty}\frac{z^{-m/2}}{m!}\,.\end{aligned}$$

Since the third sum in the last equality represents the reciprocal of the double factorial sequence we get,

$$2\sum_{n=0}^{\infty}\frac{z^{-n}}{(2n)!}=\sum_{m=0}^{\infty}\frac{z^{-m/2}}{m!}+\sum_{m=0}^{\infty}\frac{(-1)^m z^{-m/2}}{m!}\,.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{z^{-n}}{(2n)!}=\frac{1}{2}\Big[\sum_{m=0}^{\infty}\frac{(z^{-1/2})^m}{m!}+\sum_{m=0}^{\infty}\frac{(-z^{-1/2})^m}{m!}\Big]=\frac{1}{2}\Big(e^{z^{-\frac{1}{2}}}+e^{-z^{-\frac{1}{2}}}\Big)\,,$$

and the corresponding transform pair is given by

$$\boxed{\frac{u(n)}{(2n)!}\leftrightarrow\cosh(z^{-\frac{1}{2}})}\tag{3.5}$$

As a final example in this section, we determine the $\mathcal{Z}$-transform of the sequence defined by $u(n-1)/n$ making use of the division by $n+a$ and backward shift properties whose transform pairs correspond, respectively, to Eqs. (1.11) and (1.16). Let $a=0$, then

$$\begin{aligned}\mathcal{Z}\Big\{\frac{u(n-1)}{n}\Big\}&=-z^0\int_0^z\zeta^{-0-1}\mathsf{U}(\zeta-1)d\zeta=-\int_0^z\frac{1}{\zeta}\frac{\zeta^{1-1}}{\zeta-1}d\zeta=-\int_0^z\frac{d\zeta}{\zeta(\zeta-1)}\\&=-\Big(\int_0^z\frac{d\zeta}{\zeta-1}-\int_0^z\frac{d\zeta}{\zeta}\Big)=\ln(z)-\ln(z-1)\,,\end{aligned}$$

where, in the last equality, singularities are avoided considering that $|z|>1$. In this case, the transform pair has the expression,

$$\boxed{\frac{u(n-1)}{n}\leftrightarrow\ln\Big(\frac{z}{z-1}\Big)}\tag{3.6}$$

4 Transform Inversion Methods

For some applications such as the solution of difference equations it is mandatory to find a sequence φ from its transform Φ. In other words, given the $\mathcal{Z}$-transform as a function of z, i.e., $\Phi(z)$, find the corresponding sequence of n, $\varphi(n)$, such that $\mathcal{Z}\{\varphi(n)\}=\Phi(z)$. This type of problem is commonly known as *inversion* or *inverse transformation* for which there are three forms of accomplishing it as described next. It is customary to write $\varphi(n)=\mathcal{Z}^{-1}\{\Phi(z)\}$.

a) Partial Fraction development. Given $\Phi(z)$, we develop the expression $\Phi(z)/z$ as a sum of partial fractions. Then, realizing the product of z with each partial fraction it is possible to identify, using a table of direct transforms in z, the corresponding entry as a function of n and finally, $\varphi(n)$ can be obtained by linearity considering all developed fractional terms.

b) Complex Residues determination. From the definition of the inverse $\mathcal{Z}$-transform, see Eq. (1.5), we have that

$$\varphi(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \Phi(z) z^{n-1} dz \,.$$

Recalling that Cauchy's residue theorem establishes the relationship given by,

$$\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = 2\pi i \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} [\Phi(z) z^{n-1}] \,,$$

we can calculate φ by evaluating the following sum,

$$\varphi(n) = \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} [\Phi(z) z^{n-1}] \,, \tag{4.1}$$

where z_k is a pole of $\Phi(z) z^{n-1}$ for $k \in \{1, \ldots, m\}$.

c) Power Series expansion. The transform function $\Phi(z)$ is expanded in a power series of z^{-1} using the known procedure of long division and the coefficient of z^{-n} is associated with $\varphi(n)$. The previous technique does not provide a direct way for obtaining the algebraic expression for φ; however, it is practical in case that only a few terms of the sequence are needed. The power series so obtained corresponds to a Laurent series.

We remark that the first two forms of inversion give closed expressions for $\varphi(n)$. Next a few simple examples are provided of the three forms just mentioned for finding inverse $\mathcal{Z}$-transforms.

4.1 Inversion of rational functions of z

Example 4.1 Find the inverse transform of,

$$\boxed{\Phi(z) = \frac{z}{z^2 - 6z + 8}} \tag{4.2}$$

a) By developing in partial fractions we have that,

$$\frac{\Phi(z)}{z} = \frac{1}{z^2 - 6z + 8} = \frac{1}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4},$$

where, $A = (z{-}2)\frac{\Phi(z)}{z}\Big|_{z=2} = \frac{1)}{z-4}\Big|_{z=2} = -\frac{1}{2}$; $B = (z{-}4)\frac{\Phi(z)}{z}\Big|_{z=4} = \frac{1)}{z-2}\Big|_{z=4} = \frac{1}{2}$,

$$\Rightarrow \Phi(z) = -\frac{1}{2}\Big(\frac{z}{z-2} - \frac{z}{z-4}\Big) = \frac{1}{2}\Big(\frac{z}{z-4}\Big) - \frac{1}{2}\Big(\frac{z}{z-2}\Big)$$

$$\begin{aligned}\Rightarrow \mathcal{Z}^{-1}\{\Phi(z)\} &= \mathcal{Z}^{-1}\Big\{\frac{1}{2}\Big(\frac{z}{z-4}\Big) - \frac{1}{2}\Big(\frac{z}{z-2}\Big)\Big\} = \frac{1}{2}\mathcal{Z}^{-1}\Big\{\frac{z}{z-4}\Big\} - \frac{1}{2}\mathcal{Z}^{-1}\Big\{\frac{z}{z-2}\Big\} \\ &= \frac{1}{2}4^n u(n) - \frac{1}{2}2^n u(n) = \frac{1}{2}2^n(2^n - 1)u(n)\,.\end{aligned}$$

from which the result follows, i. e.,

$$\boxed{\varphi(n) = \mathcal{Z}^{-1}\Big\{\frac{z}{z^2 - 6z + 8}\Big\} = 2^{n-1}(2^n - 1)u(n)} \tag{4.3}$$

b) the residues ($m = 2$) for the given z function are determined from Eq. (4.1) as shown below,

$$\begin{aligned}\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z z^{-1}}{z^2 - 6z + 8} z^n dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z^n dz}{z^2 - 6z + 8} \\ &= \frac{1}{2\pi i}\Big(2\pi i \sum_{k=1}^{2} \underset{z=z_k}{\text{res}}\Big[\frac{z^n}{z^2 - 6z + 8}\Big]\Big) = \sum_{k=1}^{2} \underset{z=z_k}{\text{res}}\Big[\frac{z^n}{(z-2)(z-4)}\Big] \\ &= \frac{z^n}{z-4}\Big|_{z=2} + \frac{z^n}{z-2}\Big|_{z=4} = \Big(-\frac{1}{2}2^n + \frac{1}{2}4^n\Big)u(n) = 2^{n-1}(2^n - 1)u(n)\,.\end{aligned}$$

c) to obtain the power series expansions in z^{-1} we make use of the long division procedure to obtain a few terms from which the sequence pattern can be inferred. Thus,

$$
\begin{array}{r|l}
 & \underline{z^{-1} + 6z^{-2} + 28z^{-3} + 120z^{-4} + \ldots} \\
z^2 - 6z + 8 & z \\
 & \underline{-z + 6 + 8z^{-1}} \\
 & 6 - 8z^{-1} \\
 & \underline{-6 + 36z^{-1} - 48z^{-2}} \\
 & 28z^{-1} - 48z^{-2} \\
 & \underline{-28z^{-1} + 168z^{-2} - 224z^{-3}} \\
 & 120z^{-2} - 224z^{-3} \\
 & \underline{-120z^{-2} + 720z^{-3} - 960z^{-4}} \\
 & \ldots
\end{array}
$$

or, equivalently,

$$
\Phi(z) = \frac{z}{z^2 - 6z + 8} = \frac{1}{z} + \frac{6}{z^2} + \frac{28}{z^3} + \frac{120}{z^4} + \ldots\,, \tag{4.4}
$$

and it is not difficult to see that the series coefficients can be rewritten as the elements of the sequence $\varphi(n)$, where n denotes the nth-power of z^{-1}, i. e.,

$$
\begin{aligned}
&\varphi(1) = 1 = 2^{1-1}(2-1), \quad \varphi(2) = 6 = 2^{2-1}(4-1), \\
&\varphi(3) = 28 = 2^{3-1}(8-1), \ \varphi(4) = 120 = 2^{4-1}(16-1), \ldots
\end{aligned}
$$

Example 4.2 Find the inverse transform of,

$$
\boxed{\Phi(z) = \frac{3z^2 - z}{(z-1)(z-2)^2}} \tag{4.5}
$$

a) Developing in partial fractions we obtain,

$$
\begin{gathered}
\frac{\Phi(z)}{z} = \frac{3z - 1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \\
\Rightarrow A(z-2)^2 + B(z-1)(z-2) + C(z-1) = 3z - 1\,.
\end{gathered}
$$

Therefore, equating coefficients of the same z power between the left and right sides of the last expression, the values of the constants A, B, and C are readily found. Thus,

$$\begin{aligned}
z = 1 &\Rightarrow A(1-2)^2 + B(1-1)(1-2) + C(1-1) = 3(1) - 1\,,\\
&\Rightarrow A(-1)^2 = 2 \Rightarrow A = 2\,,\\
z = 2 &\Rightarrow A(2-2)^2 + B(2-1)(2-2) + C(2-1) = 3(2) - 1\,,\\
&\Rightarrow C(1)^2 = 5 \Rightarrow C = 5\,,\\
z = 0 &\Rightarrow A(0-2)^2 + B(0-1)(0-2) + C(0-1) = 3(0) - 1\,,\\
&\Rightarrow 4A + 2B - C = -1 \Rightarrow B = -2\,.
\end{aligned}$$

So, we have that,

$$\Phi(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{5z}{(z-2)^2}$$

$$\begin{aligned}
\Rightarrow \varphi(n) &= \mathcal{Z}^{-1}\Big\{\frac{2z}{z-1} - \frac{2z}{z-2} + \frac{5z}{(z-2)^2}\Big\}\\
&= 2\mathcal{Z}^{-1}\Big\{\frac{z}{z-1}\Big\} - 2\mathcal{Z}^{-1}\Big\{\frac{z}{z-2}\Big\} + 5\mathcal{Z}^{-1}\Big\{\frac{z}{(z-2)^2}\Big\}\\
&= 2u(n) - 2[2^n u(n)] + 5[n2^{n-1}u(n)] = 2u(n) - 2^{n+1}u(n) + 5n2^{n-1}u(n)\,,
\end{aligned}$$

from which the result follows, i. e.,

$$\boxed{\varphi(n) = \mathcal{Z}^{-1}\Big\{\frac{3z^2 - z}{(z-1)(z-2)^2}\Big\} = [2^{n-1}(5n-4) + 2]u(n)} \tag{4.6}$$

b) once the residues of $\Phi(z)$ are found for $m = 2$, the algebraic form of $\varphi(n)$ is readily obtained. Specifically,

$$\begin{aligned}
\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z)z^{n-1}dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{(3z-1)zz^{-1}}{(z-1)(z-2)^2}z^n dz = \sum_{k=1}^{2} \operatorname*{res}_{z=z_k}\Big[\frac{(3z-1)z^n}{(z-1)(z-2)^2}\Big]\\
&= \frac{(3z-1)z^n}{(z-2)^2}\Big|_{z=1} + \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z-2)^2\Big[\frac{(3z-1)z^n}{(z-1)(z-2)^2}\Big]\Big|_{z=2}\\
&= \frac{(2)1^n}{(-1)^2}u(n) + \frac{d}{dz}\Big[\frac{(3z-1)z^n}{(z-1)}\Big]\Big|_{z=2}\\
&= 2u(n) + \Big[\frac{(z-1)[(3z-1)z^n]' - (3z-1)z^n(z-1)'}{(z-1)^2}\Big]\Big|_{z=2}\\
&= 2u(n) + \Big[\frac{(3z-1)nz^{n-1} + 3z^n]}{z-1} - \frac{(3z-1)z^n}{(z-1)^2}\Big]\Big|_{z=2}\\
&= 2u(n) + [5n2^{n-1} + 3\cdot 2^n - 5\cdot 2^n]u(n) = [2 - 2^{n+1} + 5n2^{n-1}]u(n)\,,
\end{aligned}$$

a result that matches the expression given before in Eq. (4.6).

c) in this case the denominator is given by the polynomial,

$$(z-1)(z^2-4z+4) = (z^3-4z^2+4z)-(z^2-4z+4) = z^3-5z^2+8z-4\,.$$

Again, to obtain the power series expansions in z^{-1} we make use of the long division procedure to obtain a few terms from which the sequence pattern can be inferred. Therefore,

$$\begin{array}{r|l}
 & \underline{3z^{-1}+14z^{-2}+46z^{-3}+130z^{-4}+\ldots} \\
z^3-5z^2+8z-4 & 3z^2-z \\
 & \underline{-3z^2+15z-24+12z^{-1}} \\
 & \qquad 14z-24+12z^{-1} \\
 & \qquad \underline{-14z+70-112z^{-1}+56z^{-2}} \\
 & \qquad\qquad 46-100z^{-1}+56z^{-2} \\
 & \qquad\qquad \underline{-46+230z^{-1}-368z^{-2}+184z^{-3}} \\
 & \qquad\qquad\qquad 130z^{-1}-312z^{-2}+184z^{-3} \\
 & \qquad\qquad\qquad \underline{-130z^{-1}+650z^{-2}-1040z^{-3}+520z^{-4}} \\
 & \qquad\qquad\qquad\qquad \ldots
\end{array}$$

or, equivalently,

$$\Phi(z) = \frac{3z^2-z}{(z-1)(z-2)^2} = \frac{3}{z}+\frac{14}{z^2}+\frac{46}{z^3}+\frac{130}{z^4}+\ldots\,. \tag{4.7}$$

Consequently it can be observed that the series coefficients can be rewritten as the elements of the sequence $\varphi(n)$, as follows,

$$\varphi(1) = 3 = 2-2^{1+1}+5(1)2^{1-1}, \quad \varphi(2) = 14 = 2-2^{2+1}+5(2)2^{2-1},$$
$$\varphi(3) = 46 = 2-2^{3+1}+5(3)2^{3-1}, \ \varphi(4) = 130 = 2-2^{4+1}+5(4)2^{4-1}, \ldots$$

In Examples 4.1 and 4.2 we may combine the results obtained either by method a) or b) with c) in order to write the corresponding δ representation as expansions of the form, $\varphi(n) = \sum_{m=0}^{\infty}\varphi(m)\delta(n-m)$ where, in both cases, $\varphi(0) = 0$. Specifically,

$$2^{n-1}(2^n-1)u(n) = \delta(n-1)+6\delta(n-2)+28\delta(n-3)+120\delta(n-4)+\ldots\,,$$

and,

$$(2-2^{n+1}+5n2^{n-1})u(n) = 3\delta(n-1)+14\delta(n-2)+46\delta(n-3)+130\delta(n-4)+\dots.$$

Furthermore, according with the third method of power series expansion, if

$$\Phi(z) = \sum_{m=0}^{\infty} c_m z^{-m} \Rightarrow \mathcal{Z}^{-1}\{\Phi(z)\} = \mathcal{Z}^{-1}\Big\{\sum_{m=0}^{\infty} c_m z^{-m}\Big\}$$
$$= \sum_{m=0}^{\infty} c_m \mathcal{Z}^{-1}\{z^{-m}\} = \sum_{m=0}^{\infty} c_m \delta(n-m) = \varphi(n)\,,$$

the coefficients c_m are those determined in the developed Laurent series, that is to say,

$$c_m = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\Phi(z)}{z^{m+1}} dz \tag{4.8}$$

Example 4.3 Find the inverse transform of,

$$\boxed{\Phi(z) = \frac{z^4+4z^2+z}{z^4+z^2} = \frac{z(z^3+4z+1)}{z^2(z^2+1)}} \tag{4.9}$$

a) By developing in partial fractions we have that,

$$\frac{\Phi(z)}{z} = \frac{z^3+4z+1}{z^2(z^2+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{z^2+1}$$
$$\Rightarrow Az(z^2+1) + B(z^2+1) + (Cz+D)z^2 = z^3+4z+1 \quad \text{or}$$
$$Az^3 + Az + Bz^2 + B + Cz^3 + Dz^2 = z^3+4z+1\,.$$

Thus, $A+C=1$, $B+D=0$, $A=4$, and $B=1$; using the last two values in the first two equations we obtain, $C=-3$ and $D=-1$. Consequently,

$$\Phi(z) = 4 + \frac{1}{z} - \frac{3z^2}{z^2+1} - \frac{z}{z^2+1}$$
$$\Rightarrow \varphi(n) = \mathcal{Z}^{-1}\{4\} + \mathcal{Z}^{-1}\Big\{\frac{1}{z}\Big\} - 3\mathcal{Z}^{-1}\Big\{\frac{z^2}{z^2+1}\Big\} - \mathcal{Z}^{-1}\Big\{\frac{z}{z^2+1}\Big\}\,.$$

If we let $\omega = \pi/2$ in Eqs. (2.8) and (2.9), we get the transform pairs for the 3rd and 4th inversion terms,

$$\cos\frac{n\pi}{2}u(n) \leftrightarrow \frac{z^2}{z^2+1} \quad \text{and} \quad \sin\frac{n\pi}{2}u(n) \leftrightarrow \frac{z}{z^2+1}\,.$$

Therefore, the final expression for the sequence $\varphi = \mathcal{Z}^{-1}\{\Phi\}$ is given by,

$$\begin{aligned}\varphi(n) &= \mathcal{Z}^{-1}\left\{\frac{z^3+4z+1}{z^2(z^2+1)}\right\} \\ &= 4\delta(n) + \delta(n-1) - \left[3\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right]u(n)\end{aligned} \tag{4.10}$$

b) for the given z function the residues ($m = 3$) are calculated using Eq. (4.1) as shown next,

$$\begin{aligned}\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z^3+4z+1}{z^2(z+i)(z-i)} z^n dz \\ &= \sum_{k=1}^{3} \operatorname*{res}_{z=z_k}\left[\frac{z^3+4z+1}{z^2(z+i)(z-i)} z^n\right] \\ &= \frac{d}{dz}\left[\frac{z^n(z^3+4z+1)}{z^2+1}\right]\Big|_{z=0} + \frac{z^n(z^3+4z+1)}{(z^4+z^2)'}\Big|_{z=-i} + \frac{z^n(z^3+4z+1)}{(z^4+z^2)'}\Big|_{z=i},\end{aligned} \tag{4.11}$$

where $z = 0$ is a double pole, whereas $z = \pm i$ are two simple (imaginary) poles. First, we have that,

$$\begin{aligned}\frac{d}{dz}\left[\frac{z^n(z^3+4z+1)}{z^2+1}\right] &= \frac{(z^2+1)[nz^{n-1}(z^3+4z+1) + z^n(3z^2+4)] - z^n(z^3+4z+1)(2z)}{(z^2+1)^2} \\ &= \frac{z^{n-1}(z^2+1)[n(z^3+4z+1) + z(3z^2+4)] - 2z^{n+1}(z^3+4z+1)}{(z^2+1)^2}.\end{aligned}$$

From the last expression it turns out that for any $n > 1$ the derivative vanishes if $z = 0$. However, we can see that,

$$\lim_{n,z\to 0} \frac{d}{dz}\left[\frac{z^n(z^3+4z+1)}{z^2+1}\right] = 4 \quad \text{and} \quad \frac{d}{dz}\left[\frac{z^n(z^3+4z+1)}{z^2+1}\right]\Big|_{n=1,z=0} = 1\,.$$

The residues of the two simple poles give the following pair of complex conjugate values,

$$\begin{aligned}\frac{z^n(z^3+4z+1)}{4z^3+2z}\Big|_{z=-i} &= \frac{(-i)^3+4(-i)+1}{4(-i)^3+2(-i)}(-i)^n = \frac{1-3i}{2}(-i)^{n+1} = \frac{1-3i}{2}e^{-i\frac{\pi}{2}(n+1)} \\ \frac{z^n(z^3+4z+1)}{4z^3+2z}\Big|_{z=+i} &= \frac{i^3+4i+1}{4i^3+2i}i^n = \frac{1+3i}{2}i^{n+1} = \frac{1+3i}{2}e^{i\frac{\pi}{2}(n+1)}\,.\end{aligned}$$

Hence, upon substitution of the calculated residues in Eq. (4.11) the corresponding sequence is given by,

$$\begin{aligned}\varphi(n) &= 4\delta(n)+\delta(n-1)+\frac{1}{2}\Big[e^{i\frac{\pi}{2}(n+1)}+e^{-i\frac{\pi}{2}(n+1)}\Big]u(n)+\frac{3i}{2}\Big[e^{i\frac{\pi}{2}(n+1)}-e^{-i\frac{\pi}{2}(n+1)}\Big]u(n)\\ &= 4\delta(n)+\delta(n-1)+\cosh\Big[i\frac{\pi}{2}(n+1)\Big]u(n)+3i\sinh\Big[i\frac{\pi}{2}(n+1)\Big]u(n)\,.\end{aligned}$$

To simplify the preceding expression we recall the equivalences between hyperbolic and trigonometric functions, i. e., from

$$\begin{aligned}\cosh\Big[i\frac{\pi}{2}(n+1)\Big] &= \cos\Big(\frac{n\pi}{2}+\frac{\pi}{2}\Big) = -\sin\frac{n\pi}{2}\sin\frac{\pi}{2} = -\sin\frac{n\pi}{2} \quad \text{and}\\ 3i\sinh\Big[i\frac{\pi}{2}(n+1)\Big] &= -3\sin(\frac{n\pi}{2}+\frac{\pi}{2}\Big) = -3\cos\frac{n\pi}{2}\sin\frac{\pi}{2} = -3\cos\frac{n\pi}{2}\,,\end{aligned}$$

we finally obtain,

$$\varphi(n) = 4\delta(n)+\delta(n-1)-\Big(3\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}\Big)u(n)\,.$$

Applying the long division procedure we get a few terms of a power series expansions in z^{-1} whose coefficients form the numerical sequence determined in a) or b). Thus,

$$\begin{array}{rl} & \underline{z^0+0z^{-1}+3z^{-2}+z^{-3}-3z^{-4}-z^{-5}+\ldots}\\ z^4+z^2| & z^4+4z^2+z\\ & \underline{-z^4-z^2}\\ & \quad 3z^2+z\\ & \quad \underline{-3z^2-3z^{-2}}\\ & \qquad z-3z^{-2}\\ & \qquad \underline{-z-z^{-1}}\\ & \qquad\quad -3z^{-2}-z^{-1}\\ & \qquad\quad \underline{3z^{-2}+3z^0}\\ & \qquad\qquad -z^{-1}+3z^0\\ & \qquad\qquad \underline{z^{-1}+z^{-3}}\\ & \qquad\qquad\quad \ldots \end{array}$$

or, equivalently,

$$\Phi(z) = \frac{z^4+4z^2+z}{z^4+z^2} = 1+\frac{3}{z^2}+\frac{1}{z^3}-\frac{3}{z^4}-\frac{1}{z^5}+\ldots\,. \tag{4.12}$$

Evaluating the sequence $\varphi(n)$ displayed in Eq. (4.10), for $n = 0, \dots, 5$, gives us the coefficients found above as seen in the following list:

$$\begin{aligned}
\varphi(0) &= 4 + 0 - (3 + 0) = 4 - 3 = 1\,,\\
\varphi(1) &= 0 + 1 - (0 + 1) = 1 - 1 = 0\,,\\
\varphi(2) &= 0 + 0 - (-3 + 0) = -(-3) = 3\,,\\
\varphi(3) &= 0 + 0 - (0 - 1) = -(-1) = 1\,,\\
\varphi(4) &= 0 + 0 - (3 + 0) = -(3) = -3\,,\\
\varphi(5) &= 0 + 0 - (0 + 1) = -(1) = -1\,.
\end{aligned}$$

Example 4.4 If $a \in \mathbb{R}$ with $a \neq 0$, find the inverse transform of,

$$\boxed{\Phi(z) = \frac{1}{z + a}} \tag{4.13}$$

a) Developing in partial fractions we obtain,

$$\frac{\Phi(z)}{z} = \frac{1}{z(z + a)} = \frac{A}{z} + \frac{B}{z + a}\,,$$

where, the constants A and B are computed as follows,

$$A = \left.\frac{1}{z + a}\right|_{z=0} = a^{-1} \quad \text{and} \quad B = \left.\frac{1}{z}\right|_{z=-a} = -a^{-1}\,,$$

from which,

$$\begin{aligned}
\Phi(z) = a^{-1} - \frac{a^{-1}z}{z - (-a)} \Rightarrow \mathcal{Z}^{-1}\{\Phi(z)\} &= a^{-1}\mathcal{Z}^{-1}\{1\} - a^{-1}\mathcal{Z}^{-1}\left\{\frac{z}{z - (-a)}\right\}\\
&= a^{-1}\delta(n) + (-a)^{n-1}u(n)\,.
\end{aligned}$$

Therefore,

$$\boxed{\varphi(n) = \frac{1}{a}[\delta(n) + (-a)^n u(n)]} \tag{4.14}$$

A shorter alternative to this method considers function $\Phi(z) = 1/(z+a)$ itself a "partial" fraction. Then, we can apply the properties already proven to find its inverse $\mathcal{Z}$-transform. Thus, if we substitute a by $-a$ in the transform pair $a^n u(n) \leftrightarrow z/(z - a)$, we get the transform pair, $(-a)^n u(n) \leftrightarrow z/(z + a)$ to which a unit backward shift is applied to obtain the desired transform pair given by,

$$(-a)^{n-1}u(n - 1) \leftrightarrow \frac{z^{-1}z}{z + a} = \frac{1}{z + a}\,.$$

Now, if we let, $\widehat{\varphi}(n) = (-a)^{n-1}u(n-1)$, denote the inverse transform of $\Phi(z)$ it is easy to verify that $\varphi(n) = \widehat{\varphi}(n)$ for all $n \in \mathbb{N}$. In particular, if $n = 0$ then $\varphi(0) = a^{-1}(1) + (-a)^{-1}(1) = a^{-1} - a^{-1} = 0 = (-a)^{-1}(0) = \widehat{\varphi}(0)$ and for $n \geq 1$, $\varphi(n) = a^{-1}(0) + (-a)^{n-1}(1) = (-a)^{n-1}(1) = \widehat{\varphi}(n)$.

b) the only residue happens at the pole $z = -a$ and for $n \geq 1$ is given by,

$$\varphi(n) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z^{n-1}}{z+a} dz = z^{n-1}\big|_{z=-a} = (-a)^{n-1}u(n-1)\,.$$

Once more, the long division procedure is applied to find with a few terms the general form of the corresponding power series expansion in z^{-1}. In this case,

$$\begin{array}{l}
\qquad\quad \underline{z^{-1} - az^{-2} + a^2 z^{-3} - a^3 z^{-4} + \ldots} \\
z + a \mid 1 \\
\qquad \underline{-1 - az^{-1}} \\
\qquad\qquad -az^{-1} \\
\qquad\qquad \underline{az^{-1} + a^2 z^{-2}} \\
\qquad\qquad\qquad a^2 z^{-2} \\
\qquad\qquad\qquad \underline{-a^2 z^{-2} - a^3 z^{-3}} \\
\qquad\qquad\qquad\qquad -a^3 z^{-3} \\
\qquad\qquad\qquad\qquad \underline{a^3 z^{-3} + a^4 z^{-4}} \\
\qquad\qquad\qquad\qquad\qquad \ldots
\end{array}$$

or, equivalently,

$$\Phi(z) = \frac{1}{z+a} = \frac{1}{z} - \frac{a}{z^2} + \frac{a^2}{z^3} - \frac{a^3}{z^4} + \ldots = \sum_{n=1}^{\infty} (-a)^{n-1} z^{-n} = \sum_{n=0}^{\infty} (-a)^{n-1} u(n-1) z^{-n}$$

$$\Rightarrow \varphi(n) = (-a)^{n-1}u(n-1)\,.$$

Example 4.5 If $a, b \in \mathbb{R}$ with $a \neq b$, find the inverse transform of,

$$\boxed{\Phi(z) = \frac{z^2}{(z-a)(z-b)}} \tag{4.15}$$

a) In this case, the partial fractions development has the form,

$$\frac{\Phi(z)}{z} = \frac{z}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}\,,$$

where, the constants A and B are determined as follows,

$$A = \frac{z}{z-b}\Big|_{z=a} = \frac{a}{a-b} \quad \text{and} \quad B = \frac{z}{z-a}\Big|_{z=b} = \frac{b}{b-a} = -\frac{b}{a-b}\,,$$

$$\Rightarrow \Phi(z) = \frac{a}{a-b}\Big(\frac{z}{z-a}\Big) - \frac{b}{a-b}\Big(\frac{z}{z-b}\Big) = \frac{1}{a-b}\Big(\frac{az}{z-a} - \frac{bz}{z-b}\Big)$$
$$\Rightarrow \varphi(n) = \frac{1}{a-b}\Big[a\mathcal{Z}^{-1}\Big(\frac{z}{z-a}\Big) - b\mathcal{Z}^{-1}\Big(\frac{z}{z-b}\Big)\Big] = \frac{1}{a-b}[aa^n u(n) - bb^n u(n)]\,,$$

or what is the same,

$$\boxed{\varphi(n) = \frac{1}{a-b}[a^{n+1} - b^{n+1}]u(n)\,.} \tag{4.16}$$

The singular case for which $b = a$ can be treated by recalling that,

$$a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + \ldots + ab^{n-1} + b^n) = (a-b)\sum_{k=0}^{n} a^{n-k}b^k\,.$$

Then, it is is not difficult to see that,

$$\lim_{b\to a}\varphi(n) = \lim_{b\to a}\sum_{k=0}^{n} a^{n-k}b^k u(n) = (n+1)a^n u(n) = (n+1)a^n u(n+1)\,,$$

since, for $n \geq 0$, $u(n+1) = u(n)$; also, $\lim_{b\to a} F(z) = z^2/(z-a)^2$. Therefore, we have established another transform pair, i. e.,

$$\boxed{(n+1)a^n u(n+1) \leftrightarrow \frac{z^2}{(z-a)^2}} \tag{4.17}$$

Another way to get the same result is to consider the transform pair obtained earlier in Eq. (2.13), $na^{n-1}u(n) \leftrightarrow z/(z-a)^2$ and let $\widehat{\varphi}(n) = na^{n-1}u(n)$. By the forward shift property, we have that $\widehat{\varphi}(n+1) \leftrightarrow z[F(z) - \widehat{\varphi}(0)]$ which simplifies to $\widehat{\varphi}(n+1) \leftrightarrow zF(z)$, because $\widehat{\varphi}(0) = 0$. Consequently, as stated before,

$$\widehat{\varphi}(n+1) = (n+1)a^n u(n+1) \leftrightarrow z\Big[\frac{z}{(z-a)^2}\Big] = \frac{z^2}{(z-a)^2}\,.$$

b) for the given z function, the residues ($m = 2$) are easily found using Eq. (4.1) as shown below. Thus, for $n \geq 0$,

$$
\begin{aligned}
\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{z}{(z-a)(z-b)} z^n dz \\
&= \sum_{k=1}^{2} \operatorname*{res}_{z=z_k}\left[\frac{z^{n+1}}{(z-a)(z-b)}\right] = \frac{z^{n+1}}{z-b}\bigg|_{z=a} + \frac{z^{n+1}}{z-a}\bigg|_{z=b} \\
&= \frac{a^{n+1}}{a-b}u(n) + \frac{b^{n+1}}{b-a}u(n) = \frac{a^{n+1}-b^{n+1}}{a-b}u(n)\,. \qquad (4.18)
\end{aligned}
$$

c) Although tiresome, the procedure of long division gives us the following terms of the corresponding power series expansion in z^{-1},

$$
\begin{array}{l|l}
 & \overline{1 + (a+b)z^{-1} + (a^2+ab+b^2)z^{-2} + (a^3+a^2b+ab^2+b^3)z^{-3} + \dots} \\
z^2 - (a+b)z + ab & z^2 \\
 & \underline{-z^{-2} + (a+b)z - ab} \\
 & \quad (a+b)z - ab \\
 & \quad \underline{-(a+b)z + (a+b)^2 - ab(a+b)z^{-1}} \\
 & \qquad (a^2+ab+b^2) - ab(a+b)z^{-1} \\
 & \qquad \underline{-(a^2+ab+b^2) + (a+b)(a^2+ab+b^2)z^{-1} - ab(a^2+ab+b^2)z^{-2}} \\
 & \qquad\qquad (a^3+a^2b+ab^2+b^3)z^{-1} - ab(a^2+ab+b^2)z^{-2} \\
 & \qquad\qquad -(a^3+a^2b+ab^2+b^3)z^{-1} + \\
 & \qquad\qquad \underline{(a+b)(a^3+a^2b+ab^2+b^3)z^{-2} - ab(a^3+a^2b+ab^2+b^3)z^{-3}} \\
 & \qquad\qquad\qquad \dots
\end{array}
$$

or,

$$
\begin{aligned}
\Phi(z) = \frac{z^2}{(z-a)(z-b)} &= 1 + \frac{a+b}{z} + \frac{a^2+ab+b^2}{z^2} + \frac{a^3+a^2b+ab^2+b^3}{z^3} + \dots \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a^{n-k}b^k\Big) z^{-n} = \sum_{n=0}^{\infty}\Big(\frac{a^{n+1}-b^{n+1}}{b-a}\Big) z^{-n}\,.
\end{aligned}
$$

From the last power series, the sequence $\varphi(n)$ given in Eq. (4.18) corresponds to the n-th coefficient of z^{-n}.

4.2 Inversion of the function $e^{1/z}$

If $\Phi(z)$ is not a rational function of z it may be possible to apply "formally" the inversion methods by interpreting them correctly. That is the case in our next example.

Example 4.6 Find the inverse transform of,

$$\boxed{\Phi(z) = e^{1/z}} \tag{4.19}$$

a) Here the quotient $\Phi(z)/z$ does not make sense as in our previous examples. However we can consider the power series expansion of $\Phi(z)$ as an "infinite" partial fractions development. Thus, we have that,

$$e^{z^{-1}} = \sum_{m=0}^{\infty} \frac{(z^{-1})^m}{m!} = \sum_{m=0}^{\infty} \frac{z^{-m}}{m!}$$

$$\Rightarrow \mathcal{Z}^{-1}\{e^{z^{-1}}\} = \sum_{m=0}^{\infty} \frac{1}{m!}\mathcal{Z}^{-1}\{z^{-m}\} = \sum_{m=0}^{\infty} \frac{1}{m!}\delta(n-m) = \frac{1}{n!}\,.$$

From this result, it turns out that the corresponding transform pair is given by,

$$\boxed{e^{1/z} \leftrightarrow \frac{u(n)}{n!}} \tag{4.20}$$

b) In this particular case, we find the sequence by evaluating the complex integral and applying Cauchy's integral theorem to each term. The following chain of equalities accomplishes our purpose,

$$\begin{aligned}
\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} e^{z^{-1}} z^{n-1}dz = \frac{1}{2\pi i}\oint_{\mathcal{C}}\Big(\sum_{m=0}^{\infty}\frac{z^{-m}}{m!}\Big)z^{n-1}dz \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}}\sum_{m=0}^{\infty}\frac{z^{n-m-1}}{m!}dz = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{m!}\oint_{\mathcal{C}} z^{n-m-1}dz \\
&= \frac{1}{2\pi i}\Big[\sum_{m=0}^{n-1}\frac{1}{m!}\oint_{\mathcal{C}} z^{n-m-1}dz + \frac{1}{n!}\oint_{\mathcal{C}}\frac{dz}{z} + \sum_{m=n+1}^{\infty}\frac{1}{m!}\oint_{\mathcal{C}} z^{n-m-1}dz\Big] \\
&= \frac{1}{2\pi i}\Big(0 + \frac{2\pi i}{n!} + 0\Big) = \frac{1}{n!}\,.
\end{aligned} \tag{4.21}$$

Since the sequence obtained is valid for $n \geq 0$ we see again that $\varphi(n) = u(n)/n!$.

c) For the given function it is unnecessary to use the long division procedure since the power series expansion is already known. Hence, it is an immediate consequence that, if $e^{1/z} = \sum_{n=0}^{\infty}(1/n!)z^{-n}$, then, $\varphi(n) = u(n)/n!$.

4.3 Inversion of an analytic function of z

In this subsection we will find the inverse $\mathcal{Z}$-transform of a function Φ that is *analytic* or *holomorphic* over the region $|z| > 0$ (the complex plane except the origin). By hypothesis, $\Phi(z)$ can be expressed as a Laurent power series about $z = 0$, i. e.,

$$\begin{aligned}\Phi(z) &= \sum_{m=-\infty}^{\infty} a_m z^m = \sum_{m=-\infty}^{0} a_m z^m + \sum_{m=1}^{\infty} a_m z^m \\ &= \sum_{n=0}^{\infty} a_{-n} z^{-n} + \sum_{m=1}^{\infty} a_m z^m = \mathcal{Z}\{\varphi(n)\} + \sum_{m=1}^{\infty} a_m z^m .\end{aligned}$$

Restricting the class of analytic functions to those whose infinite non-constant polynomial part is zero, we have that $a_m = 0$ for $m \geq 1$. In that case, $\varphi(n) = \mathcal{Z}^{-1}\{\Phi(z)\} = a_{-n}$ for any $n \in \mathbb{N}$. On the other hand, recall that the mth-coefficient in the Laurent series is given by,

$$a_m = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\Phi(\zeta)}{\zeta^{m+1}} d\zeta .$$

If we let $m = -n$, the corresponding integral, except for the name of the integration variable, equals $\varphi(n)$. Thus, we see that,

$$\varphi(n) = a_{-n} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\Phi(\zeta)}{\zeta^{-n+1}} d\zeta = \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(\zeta)\zeta^{n-1} d\zeta ,$$

is the same inversion formula introduced earlier in Eq. (1.5). Furthermore, under the given conditions imposed on Φ, it turns out that the previous explanation is another way of defining the inverse transform $\mathcal{Z}^{-1}$ of Φ *without* recurring to the Laplace transform. An alternative argument based on Cauchy's integral theorem provides us with the same result. Specifically,

$$\begin{aligned}\varphi(n) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \Phi(z) z^{n-1} dz = \frac{1}{2\pi i}\oint_{\mathcal{C}}\Big(\sum_{m=-\infty}^{\infty} a_m z^m\Big) z^{n-1} dz \\ &= \sum_{m=-\infty}^{\infty} \frac{a_m}{2\pi i}\oint_{\mathcal{C}} z^{m+n-1} dz = \sum_{m\neq -n}^{\infty} \frac{a_m}{2\pi i}\oint_{\mathcal{C}} z^{m+n-1} dz + \frac{a_{-n}}{2\pi i}\oint_{\mathcal{C}} \frac{dz}{z} = a_{-n}\,.\end{aligned}$$

4.4 Inversion of the gamma function $\Gamma(z)$

As a none trivial example we dedicate this subsection to find the inverse transform of the gamma function commonly denoted by $\Gamma(z)$. That is to say, the sequence given by $\gamma(n) = \mathcal{Z}^{-1}\{\Gamma(z)\}$ for $n \geq 0$. Since Γ is a *special function* the discussion is limited to the use of the complex residues method of finding inverse transforms. In order to make clear some onward calculations we first determine the value of the following integral

$$\oint_{\mathcal{C}} \Gamma(z) dz\,,$$

where, $\mathcal{C}$ is the contour shown in Fig. 2 enclosing all singularities or poles of $\Gamma(z)$ given by the elements of the set $\mathbb{Z}_0^- = \{z \in \mathbb{Z} : z \leq 0\}$. Instead of taking for $\Gamma(z)$ its integral representation we use the alternative definition contributed by Gauss, i. e., $\Gamma(z) = \lim_{m\to\infty} \Gamma_m(z)$, where

$$\Gamma_m(z) = \frac{m!\, m^z}{z(z+1)\cdots(z+m)} = \frac{m!\, m^z}{\prod_{j=0}^{m}(z+j)}\,. \tag{4.22}$$

Then,

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \Gamma(z) dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \Big[\lim_{m\to\infty} \Gamma_m(z)\Big] dz = \sum_{k=0}^{\infty} \operatorname*{res}_{z=-k} \Big[\lim_{m\to\infty} \Gamma_m(z)\Big]\,.$$

Notice that the residues sum is an infinite series whose general coefficient is found by evaluating the limit of $\Gamma_m(-k)$ when $m \to \infty$ and $k \in \mathbb{Z}_0^-$.

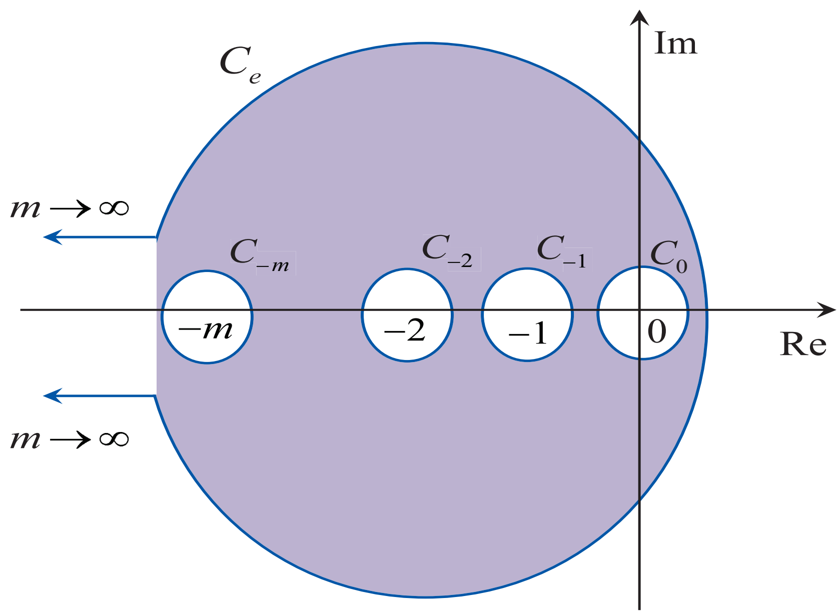

Figure 2: Contour in the $\mathbb{C}$-plane for the $\Gamma(z)$ function enclosing all its singularities.

The details are shown next,

$$\begin{aligned}
\operatorname*{res}_{z=-k}\left[\lim_{m\to\infty}\Gamma_m(z)\right] &= \lim_{m\to\infty}\left[(z+k)\frac{m!\,m^z}{\prod_{j=0}^{m}(z+j)}\Big|_{z=-k}\right] \\
&= \lim_{m\to\infty}\left[\frac{m!\,m^z}{\prod_{j=0}^{k-1}(z+j)\prod_{j=k+1}^{m}(z+j)}\Big|_{z=-k}\right] \\
&= \lim_{m\to\infty}\frac{m!\,m^{-k}}{\prod_{j=0}^{k-1}(-1)(k-j)\prod_{j=k+1}^{m}(j-k)} \\
&= \lim_{m\to\infty}\frac{m!}{m^k(-1)^k k!(m-k)!} \\
&= \lim_{m\to\infty}\frac{m(m-1)\cdots(m-k+1)(m-k)!}{m^k(-1)^k k!(m-k)!} \\
&= \lim_{m\to\infty}\frac{(-1)^k}{k!}\left[1+\frac{p(m)}{m^k}\right] = \frac{(-1)^k}{k!}\,,
\end{aligned}$$

where, $p(m)$ is a polynomial in m of degree strictly less than k.

Therefore[1],

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}\Gamma(z)dz=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=e^{-1}\Rightarrow\oint_{\mathcal{C}}\Gamma(z)dz=\frac{2\pi i}{e} \tag{4.23}$$

Now we turn our attention of finding the inverse transform of $\Gamma(z)$ using the residues method on the same contour depicted in Fig. 2. Thus, we have that,

$$\begin{aligned}\gamma(n)=\mathcal{Z}^{-1}\{\Gamma(z)\}&=\frac{1}{2\pi i}\oint_{\mathcal{C}}\Gamma(z)z^{n-1}dz\\&=\frac{1}{2\pi i}\oint_{\mathcal{C}}\Big[\lim_{m\to\infty}\Gamma_m(z)\Big]z^{n-1}dz=\sum_{k=0}^{\infty}\operatorname*{res}_{z=-k}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big]\,.\end{aligned}$$

Here, the computation of the residues must be carried in more detail since each of them depends on variable n. We first evaluate the residues for $k=0$, 1, 2, and 3 (since $z=-k$, $z=0,-1,-2,-3$) and then generalize inductively for the rest. For $z=0$, we see that,

$$\operatorname*{res}_{z=0}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big]=\operatorname*{res}_{z=0}\Big[\lim_{m\to\infty}\frac{m!\,m^z z^n}{z^2\prod_{j=1}^m(z+j)}\Big]=\lim_{m\to\infty}\frac{d}{dz}\Big(\frac{m!\,m^z z^n}{\prod_{j=1}^m(z+j)}\Big)\Big|_{z=0}\,.$$

The derivative subexpression develops into,

$$\begin{aligned}\frac{d}{dz}\Big(\frac{m!\,m^z z^n}{\prod_{j=1}^m(z+j)}\Big)&=\frac{\prod_{j=1}^m(z+j)[m!\,m^z z^n]'-m!\,m^z z^n[\prod_{j=1}^m(z+j)]'}{[\prod_{j=1}^m(z+j)]^2}\\&=\frac{m!\,m^z n z^{n-1}+m!\,m^z z^n\ln(m)}{\prod_{j=1}^m(z+j)}-\frac{m!\,m^z z^n\sum_{j=1}^m\prod_{\ell\neq j}(z+\ell)}{[\prod_{j=1}^m(z+j)]^2}\,.\end{aligned} \tag{4.24}$$

If $n=0$, the first term simplifies to

$$\frac{m!\ln(m)}{\prod_{j=1}^m j}=\frac{m!\ln(m)}{m!}=\ln(m)\,,$$

while the second term is reduced to

$$\frac{m!\sum_{j=1}^m\prod_{\ell\neq j}\ell}{[\prod_{j=1}^m j]^2}=\frac{\sum_{j=1}^m\prod_{\ell\neq j}\ell}{\prod_{j=1}^m j}=\sum_{j=1}^m\frac{\prod_{\ell\neq j}\ell}{\prod_{\ell=1}^m\ell}=\sum_{j=1}^m\frac{1}{j}\,.$$

[1]The result of Eq. (4.23) states another interesting relationship between the imaginary unit i and the irrational numbers π and e, similar to the more familiar equalities derived from Euler's identity, i.e., $e^{i\pi}=-1$ and $i^i=1/\sqrt{e^\pi}$ (a real number). See [3] for another way of calculating the same value.

Combining the two terms, the corresponding residue value at $z = 0$ for $n = 0$, turns out to be the negative of the Euler-Mascheroni constant, i. e.,

$$\lim_{m\to\infty}\Big(\ln m - \sum_{j=1}^{m}\frac{1}{j}\Big) = -\lim_{m\to\infty}\Big(\sum_{j=1}^{m}\frac{1}{j} - \ln m\Big) = -\gamma = -0.577215664901532\cdots$$

On the other hand, for $n = 1$, the first term of Eq. (4.24) simplifies to $m!/m! = 1$ and the second term equals zero. The residue value at $z = 0$ for $n = 1$ is 1 and it should be clear that if $n > 1$ both terms of Eq. (4.24) are zero. Therefore, for any $n \in \mathbb{N}$, we can write,

$$\operatorname*{res}_{z=0}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big] = -\gamma\delta(n) + \delta(n-1)\,. \tag{4.25}$$

We continue with the evaluation of the residues produced by the poles of $\Gamma(z)$ located at $z = -1, -2, -3$. The details follow,

$$\begin{aligned}
\operatorname*{res}_{z=-1}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big] &= \lim_{m\to\infty}\Big[\frac{m!\,m^z z^n}{z^2\prod_{j\neq 1}(z+j)}\Big|_{z=-1}\Big] \\
&= \lim_{m\to\infty}\frac{m!\,m^{-1}(-1)^n}{(-1)^2\,1\cdots(m-1)} = (-1)^n\,, \\
\operatorname*{res}_{z=-2}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big] &= \lim_{m\to\infty}\Big[\frac{m!\,m^z z^n}{z^2\prod_{j\neq 2}(z+j)}\Big|_{z=-2}\Big] \\
&= \lim_{m\to\infty}\frac{m!\,m^{-2}(-2)^n}{(-2)^2(-1)\,1\cdots(m-2)} = \frac{(-2)^{n-1}}{2}\,, \\
\operatorname*{res}_{z=-3}\Big[\lim_{m\to\infty}\Gamma_m(z)z^{n-1}\Big] &= \lim_{m\to\infty}\Big[\frac{m!\,m^z z^n}{z^2\prod_{j\neq 3}(z+j)}\Big|_{z=-3}\Big] \\
&= \lim_{m\to\infty}\frac{m!\,m^{-3}(-3)^n}{(-3)^2(-2)(-1)\,1\cdots(m-3)} = -\frac{(-3)^{n-1}}{6}\,.
\end{aligned}$$

By induction, we can generalize the previous pattern for any $k > 0$. Thus, we have

$$\begin{aligned}
\lim_{m\to\infty}\Big[\frac{m!\,m^{-k}(-k)^n}{(-k)^2\prod_{j\neq k}(j-k)}\Big] &= \lim_{m\to\infty}\Big[\frac{m!\,m^{-k}(-k)^{n-1}}{(-k)\prod_{j=1}^{k-1}(-1)(k-j)\prod_{j=k+1}^{m}(j-k)}\Big] \\
&= \lim_{m\to\infty}\frac{m(m-1)(m-2)\cdots(m-k+1)(m-k)!\,(-k)^{n-1}}{m^k(-1)^k k!(m-k)!} \\
&= \lim_{m\to\infty}\frac{[m^k + p(m)](-1)^{-k}(-k)^{n-1}}{m^k k!} = \frac{(-1)^k(-k)^{n-1}}{k!}\,,
\end{aligned}$$

where the last equality follows from the fact that the degree of the polynomial in m, denoted by $p(m)$, is strictly less than k; also, recall that $(-1)^{-k} = (-1)^k$ for any $k \in \mathbb{Z}^+$. Finally, by summing all residues we find the resulting expression of the inverse $\mathcal{Z}$-transform of the gamma function, i. e.,

$$\gamma(n) = -\gamma\delta(n) + \delta(n-1) + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{n-1}}{k!} u(n) \tag{4.26}$$

The elements of the gamma sequence for $n \in \{0, 1, 2, 3, 4\}$ evaluated by substitution of each n in Eq. (4.26) are listed below:

$$\gamma(0) = -\gamma + 0 + (-1)^{0-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{0-1}}{k!} = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{kk!} = -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kk!},$$

$$\gamma(1) = 0 + 1 + (-1)^{1-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{1-1}}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1},$$

$$\gamma(2) = 0 + 0 + (-1)^{2-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{2-1}}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} = e^{-1},$$

$$\gamma(3) = 0 + 0 + (-1)^{3-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{3-1}}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^k k}{(k-1)!} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}(\ell+1)}{\ell!}$$

$$= -\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(\ell+1)}{\ell!} = -\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\ell}{\ell!} - \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{(\ell-1)!} - e^{-1} = 0$$

$$\gamma(4) = 0 + 0 + (-1)^{4-1} \sum_{k=1}^{\infty} \frac{(-1)^k k^{4-1}}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k^2}{(k-1)!} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(\ell+1)^2}{\ell!}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!}(\ell^2 + 2\ell + 1) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\ell^2}{\ell!} + 2\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\ell}{\ell!} + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!}$$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\ell}{(\ell-1)!} + 2\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{(\ell-1)!} + e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(k+1)}{k!} + 2\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} + e^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}k}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - 2\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} + e^{-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} - 2e^{-1} = -e^{-1}.$$

Note that computation of $\gamma(3)$ requires a single change of summation index ($\ell = k - 1$ in the 3rd equality) whereas $\gamma(4)$ needs a couple of changes of summation index ($\ell = k - 1$ in the 3rd equality and $k = \ell - 1$ in the 7th equality). The values of the gamma sequence for $n \in \{5, 6, 7, 8, 9, 10\}$ can also be determined as previously but the details are not explicitly given since the algebraic work is more cumbersome to follow. However, the resulting values are $\gamma(5) = e^{-1}$, $\gamma(6) = 2e^{-1}$, $\gamma(7) = -9e^{-1}$, $\gamma(8) = 9e^{-1}$, $\gamma(9) = 50e^{-1}$, and $\gamma(10) = 267e^{-1}$. Looking carefully to these numerical results as well as the steps that manipulate algebraically specific summations, Eq. 4.26 can be written in a compact form as follows,

$$\begin{aligned} &\gamma(n) = -\gamma\delta(n) + \delta(n-1) + e^{-1}\alpha_n u(n) \\ &\text{where,} \quad e^{-1}\alpha_n = (-1)^{n-1}\sum_{k=1}^{\infty}\frac{(-1)^k k^{n-1}}{k!} \end{aligned} \tag{4.27}$$

For the sake of completeness, Table 1 lists the numerical values of the gamma sequence for $n = 0$ to $n = 21$. The gamma sequence (as many others) is *increasing* and for values of $n \geq 13$ it grows quite fast. We remark that the infinite series appearing in the last term of Eq. (4.26) or equivalently the term $e^{-1}\alpha_n$ in Eq. (4.27), converges for any $n \geq 0$. Symbolic computation of the infinite series in Eq. (4.26) was realized using the Maple software and numerical verification was done by approximating the same series with a finite sum of 100 terms using the Decimal Basic programming language. The reader should have in mind that the purpose of this exercise was to show, with enough detail, a nontrivial example of $\mathcal{Z}$-transform inversion.

Up to this point we have covered two fundamental aspects, a step by step exposition of a significant part of the basic properties of the $\mathcal{Z}$ transform and a careful examination of the diverse methods for finding inverse transforms. The presentation offers both an explicit and a systematic review of concepts and mathematical techniques in order that the interested reader may gain a better understanding of this topic. In the next section we treat some additional results which are interesting by themselves or otherwise are relevant in the context of practical applications.

Table 1: Values of the γ sequence and integer multiple α_n of e^{-1}

n	$\gamma(n)$	α_n
0	0.2193839	0
1	0.3678794	1
2	0.3678794	1
3	0.0000000	0
4	-0.3678794	-1
5	0.3678794	1
6	0.7357589	2
7	-3.3109150	-9
8	3.3109150	9
9	18.3939721	50
10	-98.2238108	-267
11	151.9342092	413
12	801.9771818	2180
13	-6,522.8703714	-17,731
14	18,590.0518007	50,533
15	40,531.4853106	110,176
16	-723,544.1812567	-1,996,797
17	3,656,231.9977064	9,938,669
18	-3,178,006.7503211	-8,638,718
19	-102,445,249.8212360	-278,475,061
20	934,765,660.5768400	2,540,956,509
21	-3,611,421,102.5222800	-9,816,860,358

5 Convolution and Special Sequences

5.1 Convolution of two sequences

Let $\varphi_1(n)$ and $\varphi_2(n)$ be any two sequences defined for $n \geq 0$. The *convolution product* of φ_1 and φ_2, denoted by $\varphi_1 * \varphi_2$, is a new sequence defined by,

$$(\varphi_1 * \varphi_2)(n) = \sum_{i=0}^{n} \varphi_1(i)\varphi_2(n-i) = \sum_{\substack{i,j \geq 0 \\ i+j=n}} \varphi_1(i)\varphi_2(j)\,. \tag{5.1}$$

In Eq. (5.1), the second summation follows from the first by setting $n-i=j$ and noting that any index pair (i,j) may take any combined value between 0 and n. To determine the transform, $\mathcal{Z}\{(\varphi_1 * \varphi_2)(n)\}$, Cauchy's rule of series multiplication is applied in the second equality as shown below,

$$\begin{aligned}\mathcal{Z}\{(\varphi_1 * \varphi_2)(n)\} &= \sum_{n=0}^{\infty}\Big[\sum_{i=0}^{n}\varphi_1(i)\varphi_2(n-i)\Big]z^{-n} \\ &= \Big(\sum_{n=0}^{\infty}\varphi_1(n)z^{-n}\Big)\Big(\sum_{n=0}^{\infty}\varphi_2(n)z^{-n}\Big) = \Phi_1(z)\Phi_2(z)\,.\end{aligned}$$

$$\boxed{(\varphi_1 * \varphi_2)(n)u(n) \leftrightarrow \Phi_1(z)\Phi_2(z)} \tag{5.2}$$

In words, the transform pair of Eq. (5.2), gives a simple algebraic rule: "the $\mathcal{Z}$-transform of the convolution of two sequences, φ_1 and φ_2, equals the product of their respective transforms, Φ_1 and Φ_2". Convolution is an essential operation that serves to characterize the response of a discrete linear system to a given input signal. Furthermore, one can say that convolution is an *intrinsic property* of any linear system whose physical constitutive properties may be modeled by numerical constants.

5.2 Convolution of multiple sequences

From an algebraic point of view, the set $\mathcal{S}$ of sequences $\varphi(n)$ together with the binary operations of addition $(+)$ and convolution $(*)$ has the algebraic structure of a *commutative unitary ring*. For example, since convolution is associative, i. e., $(\varphi_1 * \varphi_2) * \varphi_3 = \varphi_1 * (\varphi_2 * \varphi_3)$ for any $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}$, it is possible to write the triple convolution product without parenthesis. Furthermore, before we determine the $\mathcal{Z}$-transform of a triple convolution product we verify the following summation equality,

$$\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}}\varphi_1(i)\varphi_2(j)\varphi_3(k) = \sum_{\substack{\ell,k\geq 0\\ \ell+k=n}}\Big[\sum_{\substack{i,j\geq 0\\ i+j=\ell}}\varphi_1(i)\varphi_2(j)\Big]\varphi_3(k)\,. \tag{5.3}$$

First, we evaluate the left summation of Eq. (5.3). Let $k = p$ be a fixed but arbitrary index, then $i + j = n - p$, where $0 \leq p \leq n$. Hence, we have that,

$$\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \varphi_1(i)\varphi_2(j)\varphi_3(k) = \sum_{p=0}^{n}\Big[\sum_{\substack{i,j\geq 0\\ i+j=n-p}} \varphi_1(i)\varphi_2(j)\Big]\varphi_3(p)\,. \tag{5.4}$$

Second, to evaluate the right summation of Eq. (5.3), let $\ell = n-p$. Therefore,

$$\begin{aligned}\sum_{\substack{\ell,k\geq 0\\ \ell+k=n}}\Big[\sum_{\substack{i,j\geq 0\\ i+j=\ell}} \varphi_1(i)\varphi_2(j)\Big]\varphi_3(k) &= \sum_{\substack{\ell,p\geq 0\\ p=n-\ell}}\Big[\sum_{\substack{i,j\geq 0\\ i+j=n-p}} \varphi_1(i)\varphi_2(j)\Big]\varphi_3(p)\\ &= \sum_{p=0}^{n}\Big[\sum_{\substack{i,j\geq 0\\ i+j=n-p}} \varphi_1(i)\varphi_2(j)\Big]\varphi_3(p)\,.\end{aligned} \tag{5.5}$$

With the help of Eq. (5.3), now we find the $\mathcal{Z}$-transform of the convolution of three sequences as shown next. The last step is a consequence of function multiplication associativity.

$$\begin{aligned}\mathcal{Z}\{(\varphi_1 * \varphi_2 * \varphi_3)(n)\} &= \sum_{n=0}^{\infty}\Big[\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \varphi_1(i)\varphi_2(j)\varphi_3(k)\Big]z^{-n}\\ &= \sum_{n=0}^{\infty}\Big[\sum_{\substack{\ell,k\geq 0\\ \ell+k=n}}\Big\{\sum_{\substack{i,j\geq 0\\ i+j=\ell}} \varphi_1(i)\varphi_2(j)\Big\}\varphi_3(k)\Big]z^{-n}\\ &= \sum_{n=0}^{\infty}\Big[\sum_{\substack{i,j\geq 0\\ i+j=n}} \varphi_1(i)\varphi_2(j)\Big]z^{-n}\sum_{n=0}^{\infty}\varphi_3(n)z^{-n}\\ &= \Big(\sum_{n=0}^{\infty}\varphi_1(n)z^{-n}\sum_{n=0}^{\infty}\varphi_2(n)z^{-n}\Big)\sum_{n=0}^{\infty}\varphi_3(n)z^{-n} = \Phi_1(z)\Phi_2(z)\Phi_3(z)\,.\end{aligned}$$

The same argument used to obtain Eqs. (5.4) and (5.5) can be extended, although more laboriously, to find the transform of the convolution of a finite number of sequences. In fact, we have that,

$$\mathcal{Z}^{-1}\Big\{\prod_{k=1}^{m}\Phi_k(z)\Big\} = \sum_{\substack{\forall k, i_k\geq 0\\ i_1+\ldots+i_m=n}}\prod_{k=1}^{m}\varphi_k(i_k) \tag{5.6}$$

5.3 Self-Convolution of a sequence

An immediate consequence derived from Eq. (5.6) gives us a new transform pair stated next.

Self-Convolution of Order m. Given a sequence $\varphi(n)$ defined for $n \geq 0$, the $\mathcal{Z}$-transform of the convolution of φ with itself repeated m times, i. e., $\mathcal{Z}\{(\varphi * \cdots * \varphi)(n)\}$ is given by the transform pair,

$$\sum_{\substack{\forall k, i_k \geq 0 \\ i_1 + \ldots + i_m = n}} \left[\prod_{k=1}^{m} \varphi(i_k)\right] u(n) = (\underbrace{\varphi * \cdots * \varphi}_{m \text{ times}})(n) u(n) \leftrightarrow \Phi^m(z) \tag{5.7}$$

5.4 The modular sequence

Let $\varphi(n)$ for $n \geq 0$ be a discrete sequence whose transform is given by $\Phi(z)$. If r and m denote non-negative integers with $r \geq 0$ and $m \geq 2$, then the modular sequence of *residue* r and *modulus* m is defined by,

$$\psi_m^r(n) = \begin{cases} \varphi(n) \Leftrightarrow n \equiv r (\operatorname{mod} m) \\ \quad 0 \quad \Leftrightarrow n \not\equiv r (\operatorname{mod} m) \end{cases} \tag{5.8}$$

We are interested in finding the $\mathcal{Z}$-transform of the modular sequence in terms of $\Phi(z)$.

Modular Sequence. If $\omega = e^{2\pi i/m}$ denotes the m-th complex root of unity, then the transform pair corresponding to Eq. (5.8) is given by,

$$\psi_m^r(n) u(n) \leftrightarrow \frac{1}{m} \sum_{j=0}^{m-1} \omega^{-jr} \Phi(\omega^{-j} z) \tag{5.9}$$

The argument to verify the transformation goes as follows. Since, $\Phi(z) = \sum_{n=0}^{\infty} \varphi(n) z^{-n}$, we have that,

$$\Phi(\omega^{-j} z) = \sum_{n=0}^{\infty} \varphi(n) (\omega^{-j} z)^{-n} = \sum_{n=0}^{\infty} \varphi(n) \omega^{jn} z^{-n} .$$

Therefore,

$$\frac{1}{m}\sum_{j=0}^{m-1}\omega^{-jr}\Phi(\omega^{-j}z) = \frac{1}{m}\sum_{j=0}^{m-1}\omega^{-jr}\Big(\sum_{n=0}^{\infty}\varphi(n)\omega^{jn}z^{-n}\Big)$$
$$= \sum_{n=0}^{\infty}\varphi(n)\Big(\frac{1}{m}\sum_{j=0}^{m-1}\omega^{j(n-r)}\Big)z^{-n} = \sum_{n=0}^{\infty}\varphi(n)\sigma_m^r(n)z^{-n}\,,$$

where, the inner sum of the second equality has been abbreviated for convenience as $\sigma_m^r(n)$. From Eq. (5.8), the value of $\sigma_m^r(n)$ must be realized for each case. First, if $n \equiv r(\mathrm{mod}\, m)$ then m divides $n-r$, i. e., $(n-r)/m = k \in \mathbb{Z}^+$, and

$$\sigma_m^r(n) = \frac{1}{m}\sum_{j=0}^{m-1}e^{2\pi ij\left(\frac{n-r}{m}\right)} = \frac{1}{m}\sum_{j=0}^{m-1}e^{(2\pi k)ij} = \frac{1}{m}\sum_{j=0}^{m-1}e^{2\pi ij}\,.$$

The last equality is a consequence of e^z being a periodic function. Thus,

$$\sigma_m^r(n) = \frac{1}{m}\Big[1 + e^{2\pi i} + e^{4\pi i} + \cdots + e^{2(m-1)\pi i}\Big] = 1\,.$$

Second, if $n \not\equiv r(\mathrm{mod}\, m)$, then $(n-r)/m$ is not an integer. Hence, in this case the resulting value turns to be the sum of a geometric progression, i. e.,

$$\sigma_m^r(n) = \frac{1}{m}\sum_{j=0}^{m-1}e^{2\pi ij\left(\frac{n-r}{m}\right)} = \frac{1}{m}\sum_{j=0}^{m-1}\Big[e^{2\pi i\left(\frac{n-r}{m}\right)}\Big]^j = \frac{1}{m}\frac{e^{2\pi i\left(\frac{n-r}{m}\right)m}-1}{e^{2\pi i\left(\frac{n-r}{m}\right)}-1} = 0\,.$$

Furthermore, recalling that the integers module m are given by the set $\mathbb{Z}_m = \{[r] : 0 \le r \le m-1\}$, we can see that $\sigma_m^r(n) = \chi_{[r]}$, where $\chi_{[r]}$ is the characteristic function defined with respect to the residual class $[r] \in \mathbb{Z}_m$. Consequently,

$$\sum_{n=0}^{\infty}\varphi(n)\sigma_m^r(n)z^{-n} = \sum_{n=0}^{\infty}\varphi(n)\chi_{[r]}z^{-n} = \sum_{n=0}^{\infty}\psi_m^r(n)z^{-n}\,.$$

Some particular sequences whose pattern is based on the modular sequence are listed below together with its corresponding transform pair.

Example 5.1 Find the $\mathcal{Z}$-transform of the modular sequences:

$$\{\varphi(0), \varphi(2), \varphi(4), \varphi(6), \ldots\}\,, \quad \{\varphi(1), \varphi(3), \varphi(5), \varphi(7), \ldots\}\,, \text{ and } \quad \{\varphi(0), \varphi(3), \varphi(6), \varphi(9), \ldots\}\,.$$

$$\{\varphi(2k)\}_{k\in\mathbb{N}} = \psi_2^0(n) \leftrightarrow \tfrac{1}{2}\sum_{j=0}^{1} \Phi(e^{-\pi i j} z) = \tfrac{1}{2}[\Phi(z) + \Phi(-z)]\,,$$

$$\{\varphi(2k+1)\}_{k\in\mathbb{N}} = \psi_2^1(n) \leftrightarrow \tfrac{1}{2}\sum_{j=0}^{1} e^{-\pi i j}\Phi(e^{-\pi i j} z) = \tfrac{1}{2}[\Phi(z) - \Phi(-z)]\,,$$

$$\{\varphi(3k)\}_{k\in\mathbb{N}} = \psi_3^0(n) \leftrightarrow \tfrac{1}{3}\sum_{j=0}^{2} \Phi(e^{-2\pi i j/3} z)\,, \text{ and}$$

$$= \tfrac{1}{3}\Big[\Phi(z) + \Phi\Big(\big[-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\big]z\Big) + \Phi\Big(\big[-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\big]z\Big)\Big]\,.$$

Example 5.2 Find the $\mathcal{Z}$-transform of the modular sequence:

$$\{1/0!, 1/4!, 1/8!, 1/12!, 1/16!, \ldots\}\,.$$

We use the transform pair, $u(n)/n! \leftrightarrow e^{1/z}$ (see Eq. (3.4)), together with the modular transform pair,

$$\{(4k)!\}_{k\in\mathbb{N}} = \psi_4^0(n) \leftrightarrow \tfrac{1}{4}\sum_{j=0}^{3} \Phi(e^{-\pi i j/2} z) = \tfrac{1}{4}[\Phi(z) + \Phi(-iz) + \Phi(-z) + \Phi(iz)]\,,$$

to obtain the result. Thus,

$$\mathcal{Z}\{\{(4k)!\}_{k\in\mathbb{N}}\} = \tfrac{1}{4}[e^{1/z} + e^{-1/z} + e^{i/z} + e^{-i/z}] = \tfrac{1}{4}\big[\cosh\big(\tfrac{1}{z}\big) + \cos\big(\tfrac{1}{z}\big)\big]$$

In the last step of the previous expression we use the identity, $\cosh(iz) = \cos(z)$.

5.5 The harmonic sequence

Recall that the n-th *harmonic number* is defined by the following finite sum for any $n \geq 1$, i.e.,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \tag{5.10}$$

We wish to find the $\mathcal{Z}$-transform of the sequence $\varphi(n) = H_n$. If in Eq. (5.10) we let $n = 0$ then we can set $H_0 = 0$ without problem. Thus, using the transform pair for the sequence formed by summing the elements of another sequence (cf. Eq. (3.3)) combined with the transform pair established in Eq. (3.6), we see that,

$$\sum_{k=1}^{n} \varphi(k) \leftrightarrow \left(\frac{z}{z-1}\right)\Phi(z) \Rightarrow \mathcal{Z}\{H_n\} = \mathcal{Z}\left\{\sum_{k=1}^{n} \frac{1}{k}\right\} = \left(\frac{z}{z-1}\right)\mathcal{Z}\left\{\frac{u(n-1)}{n}\right\}.$$

from which we obtain the following transform pair.

Harmonic Sequence. For $n \geq 1$ and $|z| > 1$,

$$H_n u(n) \leftrightarrow \frac{z}{z-1} \ln\left(\frac{z}{z-1}\right) \tag{5.11}$$

5.6 A sequence involving Bernoulli numbers

Here we consider the power series expansion of the complex function given by,

$$B(z) = \frac{z^{-1}}{e^{z^{-1}} - 1} \tag{5.12}$$

Specifically, we have that,

$$B(z) = \frac{1}{z(e^{z^{-1}} - 1)} = \frac{1}{z \sum_{m=1}^{\infty} \frac{1}{m!} z^{-m}} = \left[\sum_{m=1}^{\infty} \frac{z^{-m+1}}{m!}\right]^{-1} = \left[\sum_{n=0}^{\infty} \frac{z^{-n}}{(n+1)!}\right]^{-1}.$$

Since the leading coefficient in the series expansion does not vanish, i.e., $1/(n+1)! \neq 0$ for $n = 0$, it is possible to *invert* the given power series.

Hence, based on Cauchy's rule for power series multiplication, it is required to determine the b_n coefficients defining the inverted series in such a way that the following equality is satisfied,

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} \sum_{n=0}^{\infty} b_n z^{-n} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{b_{n-j}}{(j+1)!} z^{-n} = 1\,.$$

From the last equality, we can find the numerical values of b_n, e.g., for $n = 0, \ldots, 4$ as shown next.

$$n = 0 \Rightarrow \sum_{j=0}^{0} \frac{b_{0-j}}{(j+1)!} = b_0 = 1,$$

$$n = 1 \Rightarrow \sum_{j=0}^{1} \frac{b_{1-j}}{(j+1)!} = b_1 + \tfrac{1}{2} b_0 = 0 \Rightarrow b_1 = -\tfrac{1}{2},$$

$$n = 2 \Rightarrow \sum_{j=0}^{2} \frac{b_{2-j}}{(j+1)!} = b_2 + \tfrac{1}{2} b_1 + \tfrac{1}{6} b_0 = 0 \Rightarrow b_2 = \tfrac{1}{4} - \tfrac{1}{6} = \tfrac{1}{12},$$

$$n = 3 \Rightarrow \sum_{j=0}^{3} \frac{b_{3-j}}{(j+1)!} = b_3 + \tfrac{1}{2} b_2 + \tfrac{1}{6} b_1 + \tfrac{1}{24} b_0 = 0 \Rightarrow b_3 = -\tfrac{1}{24} + \tfrac{1}{12} - \tfrac{1}{24} = 0,$$

$$n = 4 \Rightarrow \sum_{j=0}^{4} \frac{b_{4-j}}{(j+1)!} = b_4 + \tfrac{1}{2} b_3 + \tfrac{1}{6} b_2 + \tfrac{1}{24} b_1 + \tfrac{1}{120} b_0 = 0 \Rightarrow b_4 = -\tfrac{1}{72} + \tfrac{1}{48} - \tfrac{1}{120} = -\tfrac{1}{720}.$$

It turns out that,

$$\sum_{j=0}^{n} \frac{b_{n-j}}{(j+1)!} = \frac{B_n}{n!}\,,$$

where, the B_n correspond to the sequence of Bernoulli numbers. Therefore, we get another interesting $\mathcal{Z}$-transform pair given by,

$$\boxed{\frac{B_n u(n)}{n!} \leftrightarrow \frac{z^{-1}}{e^{z^{-1}} - 1}} \tag{5.13}$$

Although outside the scope of the present work, we remark that in Eq. (5.13), the region of convergence is given by $|z| > 1/2\pi$. This last example closes our exposition of the $\mathcal{Z}$-transform going from the fundamental results and ending with other problems of a more intricate nature.

6 Ordinary Difference Equations

6.1 Transform of a second order O∆E

In this section we make use of the $\mathcal{Z}$-transform and its properties to solve elementary problems relative to ordinary linear difference equations in a similar way as it is done for solving ordinary linear differential equations with the Laplace transform. As it is customary to abbreviate an *Ordinary Differential Equation* as ODE, here we abbreviate the words *Ordinary Difference Equation* as O∆E using the Greek letter Δ to denote a mathematical finite difference. For simplicity, the following discussion is limited to the case of *ordinary linear difference equations* of the *second order* with *constant real coefficients*. The corresponding mathematical expression of such equation is given by

$$a_2\varphi(n+2) + a_1\varphi(n+1) + a_0\varphi(n) = \psi(n) \quad \text{with} \quad a_0, a_1, a_2 \in \mathbb{R}, \tag{6.1}$$

where, $\varphi(n)$ represents the *unknown sequence* or *solution sequence*, $\psi(n)$ is a given known sequence, and a_k for $k = 0, 1, 2$ are constants satisfying the relations $a_2 \neq 0 \neq a_0$ to avoid trivial cases. If the right hand member sequence in Eq. (6.1) equals zero, i. e., $\psi(n) = 0$ for all $n \geq 0$, the ordinary difference equation is said to be *homogeneous*. Note that the order of the difference equation is given by the highest "forward shift" added to the integer argument of $\varphi(n)$. From now on, we will use the subindex variable notation instead of the functional one we have used. Thus, we let $\varphi_n = \varphi(n)$. Furthermore, if the values of φ_n for $n = 0, 1$ are known, then, we can state the corresponding *initial value problem* (IVP) associated with Eq. (6.1) as follows: find a numerical sequence φ_n that solves the equation,

$$\begin{aligned} &a_2\varphi_{n+2} + a_1\varphi_{n+1} + a_0\varphi_n = \psi_n \quad \text{with} \quad a_0, a_1, a_2 \in \mathbb{R}, \\ &\text{subject to} \quad \varphi_0 = \alpha \ ; \ \varphi_1 = \beta \ ; \ \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{6.2}$$

As in the case of linear ODE's, the *complete solution* is found by adding the *complementary solution* to a *particular solution*. Recall that the complementary solution is obtained by solving the associated homogeneous ordinary difference equation, i.e., the right term in Eq. (6.2) is set to zero. In symbols,

$$\varphi_n = \varphi_n^c + \varphi_n^p \quad \forall n \in \mathbb{N}. \tag{6.3}$$

The corresponding initial conditions stated in Eq. (6.2) must satisfy Eq. (6.3). Hence, $\varphi_0 = \varphi_0^c + \varphi_0^p = \alpha$ and $\varphi_1 = \varphi_1^c + \varphi_1^p = \beta$. Applying the $\mathcal{Z}$-transform to both sides of Eq. (6.1), we see that,

$$\mathcal{Z}\{a_2\varphi_{n+2} + a_1\varphi_{n+1} + a_0\varphi_n\} = \mathcal{Z}\{\psi_n\},$$

or, by linearity, we have that,

$$a_2\mathcal{Z}\{\varphi_{n+2}\} + a_1\mathcal{Z}\{\varphi_{n+1}\} + a_0\mathcal{Z}\{\varphi_n\} = \Psi(z). \tag{6.4}$$

The terms of the left expression in Eq. (6.4) are given by,

$$a_2\mathcal{Z}\{\varphi_{n+2}\} = a_2[z^2\Phi(z) - z^2\varphi_0 - z\varphi_1] = a_2z^2\Phi(z) - a_2\alpha z^2 - a_2\beta z$$
$$a_1\mathcal{Z}\{\varphi_{n+1}\} = a_1[z\Phi(z) - z\varphi_0] = a_1z\Phi(z) - a_1\alpha z\,, \quad \text{and} \quad a_0\mathcal{Z}\{\varphi_n\} = a_0\Phi(z)\,.$$

After substitution, the same transformed equation simplifies algebraically to,

$$(a_2z^2 + a_1z + a_0)\Phi(z) = a_2\alpha z^2 + (a_2\beta + a_1\alpha)z + \Psi(z)\,.$$
$$\Rightarrow \Phi(z) = \frac{a_2\alpha z^2 + (a_2\beta + a_1\alpha)z}{a_2z^2 + a_1z + a_0} + \frac{\Psi(z)}{a_2z^2 + a_1z + a_0}\,.$$

If,

$$p(z) = z^2 + \left(\frac{a_1}{a_2}\right)z + \left(\frac{a_0}{a_2}\right) = (z - \lambda_1)(z - \lambda_2)\,,$$

denotes the second degree polynomial with roots λ_1 and λ_2 then, the $\mathcal{Z}$-transform of the unknown sequence φ_n can be expressed as,

$$\Phi(z) = \frac{\alpha z^2}{(z-\lambda_1)(z-\lambda_2)} + \left(\beta + \alpha\frac{a_1}{a_2}\right)\frac{z}{(z-\lambda_1)(z-\lambda_2)} \cdots + \frac{z}{a_2(z-\lambda_1)(z-\lambda_2)}\frac{\Psi(z)}{z} \tag{6.5}$$

6.2 Solutions of an IVP for a second order O∆E

It is now an immediate consequence of Eq. (6.5) that φ_n, the solution to the IVP posed in Eq. (6.2) can be found by applying, in linear fashion, the inverse $\mathcal{Z}$-transform to each term. Therefore, $\varphi_n = \mathcal{Z}^{-1}\{\Phi(z)\}$. However, the form of the solution will depend upon the nature of the roots λ_1 and λ_2 of the second degree *characteristic polynomial* $p(z)$. The solutions given next, in three separate cases, are based on the *discriminant* value computed as $D = a_1^2 - 4a_2a_0$.

Case I-Distinct Real Roots. Here, $D > 0$ and $\lambda_1 \neq \lambda_2 \in \mathbb{R}$. Based on Eqs. (4.15), (4.16), the forward shift in one unit and convolution properties, we have that,

$$\begin{aligned}
\mathcal{Z}^{-1}\Big\{\frac{z^2}{(z-\lambda_1)(z-\lambda_2)}\Big\} &= \frac{1}{\lambda_1-\lambda_2}(\lambda_1^{n+1}-\lambda_2^{n+1})u_n,\\
\mathcal{Z}^{-1}\Big\{\frac{z}{(z-\lambda_1)(z-\lambda_2)}\Big\} &= \frac{1}{\lambda_1-\lambda_2}(\lambda_1^{n}-\lambda_2^{n})u_{n-1},\\
\mathcal{Z}^{-1}\Big\{\frac{z}{(z-\lambda_1)(z-\lambda_2)}\frac{\Psi(z)}{z}\Big\} &= \frac{1}{\lambda_1-\lambda_2}(\lambda_1^{n}-\lambda_2^{n})u_{n-1} * \psi_{n-1}\\
&= \frac{1}{\lambda_1-\lambda_2}\sum_{k=0}^{n}(\lambda_1^{k}-\lambda_2^{k})u_{k-1}\psi_{n-1-k}\\
&= \frac{1}{\lambda_1-\lambda_2}\sum_{k=1}^{n-1}(\lambda_1^{k}-\lambda_2^{k})\psi_{n-1-k}\,.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \varphi_n^c &= \frac{1}{\lambda_1-\lambda_2}\Big[\alpha(\lambda_1^{n+1}-\lambda_2^{n+1})u_n + \Big(\beta+\alpha\frac{a_1}{a_2}\Big)(\lambda_1^{n}-\lambda_2^{n})u_{n-1}\Big]\\
\varphi_n^p &= \frac{1}{a_2(\lambda_1-\lambda_2)}\sum_{k=1}^{n-1}(\lambda_1^{k}-\lambda_2^{k})\psi_{n-1-k}
\end{aligned} \tag{6.6}$$

Also, we verify that the initial conditions are satisfied. First, $\varphi_0^c = \alpha$ since $u(-1) = 0$ and $\varphi_0^p = 0$ because the upper summation index is -1 and the sum is void. Second, $\varphi_1^c = \alpha(\lambda_1+\lambda_2)+\beta+(a_1/a_2)\alpha = \beta$ since $\lambda_1+\lambda_2 = -(a_1/a_2)$ in $p(z)$, and $\varphi_1^p = 0$ because the upper summation index is 0 and the sum is null. Therefore, $\varphi_0 = \varphi_0^c + \varphi_0^p = \alpha$ and $\varphi_1 = \varphi_1^c + \varphi_1^p = \beta$. For the sake of completeness, we check the solution for $n = 2$, i.e., $\varphi_2 = \varphi_2^c + \varphi_2^p$. Clearly,

$$a_2\varphi_2 + a_1\varphi_1 + a_0\varphi_0 = \psi_0 \Rightarrow \varphi_2 = \frac{1}{a_2}\psi_0 - \frac{a_1}{a_2}\beta - \frac{a_0}{a_2}\alpha. \tag{6.7}$$

Evaluating the complementary and particular solutions for $n = 2$, their sum simplifies to,

$$\varphi_2^c + \varphi_2^p = \alpha(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) + \Big(\beta + \frac{a_1}{a_2}\alpha\Big)(\lambda_1 + \lambda_2) + \frac{1}{a_2}\psi_0,$$

or, in terms of the sum and product of the roots λ_1 and λ_2, it is equivalent to

$$\frac{1}{a_2}\psi_0 + \alpha(\lambda_1 + \lambda_2)^2 - \alpha\lambda_1\lambda_2 + \beta(\lambda_1 + \lambda_2) + \frac{a_1}{a_2}\alpha(\lambda_1 + \lambda_2). \tag{6.8}$$

As before, from the second degree polynomial $p(z)$, we have that $\lambda_1 + \lambda_2 = -a_1/a_2$ and $\lambda_1\lambda_2 = a_0/a_2$. After substitution in Eq. (6.8) the result matches the right expression given in Eq. (6.7).

Case II-Equal Real Roots. Since $D = 0$, $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, and Eq. (6.5) simplifies to

$$\Phi(z) = \frac{\alpha z^2}{(z-\lambda)^2} + \Big(\beta + \alpha\frac{a_1}{a_2}\Big)\frac{z}{(z-\lambda)^2} + \frac{z}{a_2(z-\lambda)^2}\frac{\Psi(z)}{z}.$$

Using Eqs. (4.17) and (2.13) as well as the convolution property, we obtain the following inverse transforms,

$$\begin{aligned}
\mathcal{Z}^{-1}\Big\{\frac{z^2}{(z-\lambda)^2}\Big\} &= (n+1)\lambda^n u_{n+1}\,, \\
\mathcal{Z}^{-1}\Big\{\frac{z}{(z-\lambda)^2}\Big\} &= n\lambda^{n-1}u_n\,, \\
\mathcal{Z}^{-1}\Big\{\frac{z}{(z-\lambda)^2}\frac{\Psi(z)}{z}\Big\} &= [n\lambda^{n-1}u_n] * \psi_{n-1} \\
&= \sum_{k=0}^{n} k\lambda^{k-1}u_k\psi_{n-1-k} = \sum_{k=1}^{n-1} k\lambda^{k-1}\psi_{n-1-k}\,.
\end{aligned}$$

$$\Rightarrow \varphi_n^c = \alpha(n+1)\lambda^n u_{n+1} + \Big(\beta + \alpha\frac{a_1}{a_2}\Big) n\lambda^{n-1} u_n$$
$$\varphi_n^p = \frac{1}{a_2}\sum_{k=1}^{n-1} k\lambda^{k-1}\psi_{n-1-k} \tag{6.9}$$

To prove that the set of initial conditions is satisfied we follow a similar argument as in Case I. Thus, $\varphi_0^c = \alpha$ since $nu(n) = 0$ for $n = 0$ and $\varphi_0^p = 0$ because the upper summation index is -1 and the sum is void. Also, $\varphi_1^c = (2\lambda)\alpha + \beta + (a_1/a_2)\alpha = \beta$ because $2\lambda = -a_1/a_2$ in $p(z)$, and $\varphi_1^p = 0$ since the upper summation index is 0 and the corresponding sum is also null. So, $\varphi_0 = \varphi_0^c + \varphi_0^p = \alpha$ and $\varphi_1 = \varphi_1^c + \varphi_1^p = \beta$. Again, we verify the solution for $n = 2$, i.e., $\varphi_2 = \varphi_2^c + \varphi_2^p$. Note that Eq. (6.7) is the same for this case. Substitution of $n = 2$ in the complementary and particular solutions in Eq. (6.9) gives us

$$\varphi_2^c + \varphi_2^p = 3\alpha\lambda^2 + \Big(\beta + \frac{a_1}{a_2}\alpha\Big)(2\lambda) + \frac{1}{a_2}\psi_0,$$

which, rearranged and expanded has the form,

$$\frac{1}{a_2}\psi_0 + 3\alpha\lambda^2 + \beta(2\lambda) + \frac{a_1}{a_2}\alpha(2\lambda). \tag{6.10}$$

As before, from the second degree polynomial $p(z)$, we have that $2\lambda = -a_1/a_2$ and $\lambda^2 = a_0/a_2$. Observe that,

$$3\alpha\lambda^2 + \frac{a_1}{a_2}\alpha(2\lambda) = 3\alpha\frac{a_0}{a_2} - \alpha\Big(\frac{a_1}{a_2}\Big)^2 = -\alpha\frac{a_0}{a_2},$$

since, $D = 0$ implies that, $a_1^2 = 4a_2a_0$ or $(a_1/a_2)^2 = 4(a_0/a_2)$. Consequently, Eq. (6.10) simplifies to the right expression in Eq. (6.7).

Case III-Complex Conjugate Roots. Here, $D < 0$ and $\lambda_2 = \bar{\lambda}_1 \in \mathbb{C}$ with $\lambda_1 = \xi + i\eta$ $(\eta \neq 0)$. Equivalently, in polar form, $\lambda_1 = \rho e^{i\theta}$ where $\rho = \sqrt{\xi^2 + \eta^2}$ and $\theta = \tan^{-1}(\eta/\xi)$. Considering that $\lambda_1 - \lambda_2 = 2i\eta$, and applying Euler's identity we see that,

$$\lambda_1^{n+1} = \rho^{n+1} e^{i(n+1)\theta} = \rho^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta] \text{ and}$$
$$\lambda_2^{n+1} = \rho^{n+1} e^{-i(n+1)\theta} = \rho^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta]$$
$$\Rightarrow \lambda_1^{n+1} - \lambda_2^{n+1} = 2i\rho^{n+1}\sin(n+1)\theta.$$

Hence, the inverse $\mathcal{Z}^{-1}$-transforms of Case I applied to the complex conjugate roots yield a solution expressed as *real sequences*. Specifically,

$$\begin{aligned} \varphi_n^c &= \frac{\alpha}{\eta}\rho^{n+1}\sin[(n+1)\theta]u_n + \left(\beta + \alpha\frac{a_1}{a_2}\right)\rho^n \sin(n\theta)u_{n-1} \\ \varphi_n^p &= \frac{1}{a_2\eta}\sum_{k=1}^{n-1}\rho^k \sin(k\theta)\psi_{n-1-k} \end{aligned} \tag{6.11}$$

Once more, we see that the initial conditions are satisfied. Here, $\varphi_0^c = \alpha$ since $\sin(n\theta) = 0$ or $u(-1) = 0$ for $n = 0$ and $\rho\sin\theta = \eta$. In addition, $\varphi_0^p = 0$ because the upper summation index is -1 and the sum is void. On the other hand,

$$\varphi_1^c = 2\frac{\alpha}{\eta}\rho^2\sin\theta\cos\theta + \left(\beta + \alpha\frac{a_1}{a_2}\right)\rho\sin\theta = \alpha(2\xi) + \beta + \frac{a_1}{a_2}\alpha = \beta, \tag{6.12}$$

since $2\xi = \lambda_1 + \lambda_2 = -a_1/a_2$ in $p(z)$. Also, $\varphi_1^p = 0$ since the upper summation index is 0 and the corresponding sum is also null. Hence, $\varphi_0 = \varphi_0^c + \varphi_0^p = \alpha$ and $\varphi_1 = \varphi_1^c + \varphi_1^p = \beta$.

6.3 Examples of IVP's for second order O∆E

Some specific examples are given in this subsection to illustrate the formulas found in the previous discussion for the complete solution given a 2nd order difference equation IVP.

Example 6.1 Find the solution to the 2nd order difference equation IVP,

$$\varphi_{n+2} - \varphi_{n+1} - 6\,\varphi_n = 0 \quad \text{with} \quad \varphi_0 = \alpha = 0\ ;\ \varphi_1 = \beta = 1. \tag{6.13}$$

Note that Eq. (6.13) is an homogeneous equation since $\psi_n = 0$ for all $n \geq 0$. The coefficients are $a_2 = 1 \neq 0$, $a_1 = -1$, and $a_0 = -6 \neq 0$. Hence, $p(z) = z^2 - z - 6$ and $D = a_1^2 - 4a_2a_0 = (-1)^2 + 24 = 25 > 0$ (positive discriminant) that corresponds to Case I of distinct real roots. The two roots are given by

$$\lambda_1 = \frac{1+5}{2} = 3 \quad \text{and} \quad \lambda_2 = \frac{1-5}{2} = -2,$$

then, since $\varphi_n^p = 0$, and substituting the values of λ_1, λ_2, α, and β in Eq.(6.6), the solution to the given IVP is

$$\boxed{\varphi_n = \frac{1}{5}\Big[3^n - (-2)^n\Big] u_{n-1}}$$

Example 6.2 Find the solution to the 2nd order difference equation IVP,

$$\boxed{\varphi_{n+2} - 6\varphi_{n+1} + 9\,\varphi_n = \delta_n \quad \text{with} \quad \varphi_0 = \alpha = 5\ ;\ \varphi_1 = \beta = 12.} \tag{6.14}$$

Note that Eq. (6.14) is an heterogeneous equation since $\psi_n = \delta_n$ for all $n \geq 0$. The coefficients are $a_2 = 1 \neq 0$, $a_1 = -6$, and $a_0 = 9 \neq 0$. Thus, $p(z) = z^2 - 6z - 9$ and $D = a_1^2 - 4a_2a_0 = (-6)^2 - 36 = 0$ (zero discriminant) that corresponds to Case II of equal real roots. The double root is $\lambda = 3$, after substitution of its value as well as the given initial conditions, $\alpha = 5$ and $\beta = 12$, in Eq. (6.9), we have that

$$\begin{aligned} \varphi_n^c &= 5(n+1)3^n u_{n+1} + (12 - \tfrac{6}{1} \cdot 5)n3^{n-1}u_n \\ &= 5 \cdot 3^n (n+1) u_{n+1} + (-18) \cdot 3^{n-1} n u_n, \\ \varphi_n^p &= \sum_{k=1}^{n-1} k3^{k-1}\delta_{n-k-1} = 3^{n-2}(n-1)u_{n-1}, \end{aligned}$$

where, in φ_n^p, $\delta_{n-1-k} = 1$ if $k = n-1$ and $\delta_{n-1-k} = 0$ if $k \neq n-1$. Therefore, the solution to the given IVP is

$$\varphi_n = 5 \cdot 3^n(n+1)u_{n+1} - 18 \cdot 3^{n-1} n u_n + 3^{n-2}(n-1)u_{n-1}$$

or, equivalently,

$$\boxed{\varphi_n = 3^{n-2}[45(n+1)u_{n+1} - 54nu_n + (n-1)u_{n-1}]}$$

From the difference equation Eq. (6.14) we can compute *recursively* some values of φ_n and check them against the *non-recursive* or closed solution. Thus, for example,

$$\begin{aligned}
\varphi_2 &= 6\varphi_1 - 9\varphi_0 + 1 = \;\; 72 - \;\; 45 + 1 = 28 = 3^0(135 - 108 + 1),\\
\varphi_3 &= 6\varphi_2 - 9\varphi_1 + 0 = 168 - 108 + 0 = 60 = 3^1(180 - 162 + 2),\\
\varphi_4 &= 6\varphi_3 - 9\varphi_2 + 0 = 360 - 252 + 0 = 28 = 3^2(225 - 216 + 3), \ldots
\end{aligned}$$

Example 6.3 Find the solution to the 2nd order difference equation IVP,

$$\boxed{\varphi_{n+2} - 5\varphi_{n+1} - 6\,\varphi_n = u_n \quad \text{with} \quad \varphi_0 = \alpha = 3\;;\; \varphi_1 = \beta = 11.} \tag{6.15}$$

Again, Eq. (6.15) is an heterogeneous equation since $\psi_n = u_n$ for all $n \geq 0$. The coefficients are $a_2 = 1 \neq 0$, $a_1 = -5$, and $a_0 = -6 \neq 0$. Hence, $p(z) = z^2 - 5z - 6$ and $D = a_1^2 - 4a_2a_0 = (-5)^2 + 24 = 49 > 0$ (positive discriminant), so Case I applies. The two different roots are

$$\lambda_1 = \frac{5+7}{2} = 6 \quad \text{and} \quad \lambda_2 = \frac{5-7}{2} = -1.$$

After substitution of the root values and the given initial conditions, $\alpha = 3$ and $\beta = 11$, in Eq. (6.6), we have that

$$\begin{aligned}
\varphi_n^c &= \frac{1}{7}\Big[3(6^{n+1} - (-1)^{n+1})u_n - 4(6^n - (-1)^n)u_{n-1}\Big],\\
\varphi_n^p &= \frac{1}{7}\sum_{k=1}^{n-1}(6^k - (-1)^k)u_{n-k-1} = \frac{1}{7}\Big[\sum_{k=1}^{n-1}(6^k - (-1)^k)\Big]u_{n-2},
\end{aligned}$$

where, in φ_n^p, $u_{n-k-1} = 1$ if $n-1 \geq k$ and $u_{n-k-1} = 0$ if $n-1 < k$. Also, the last sum between square brackets simplifies to

$$\sum_{k=1}^{n-1} 6^k - \sum_{k=1}^{n-1}(-1)^k = 6\frac{6^{n-1}-1}{6-1} - (-1)\frac{(-1)^{n-1}-1}{(-1)-1} = \tfrac{6}{5}(6^{n-1}-1) + \tfrac{1}{2}((-1)^{n-1}-1).$$

Therefore, the solution to the given IVP is

$$\boxed{\varphi_n = \frac{1}{7}\Big[3(6^{n+1} - (-1)^{n+1})u_n - 4(6^n - (-1)^n)u_{n-1} + \tfrac{6}{5}(6^{n-1}-1) + \tfrac{1}{2}((-1)^{n-1}-1)\Big]}$$

Below are displayed some calculated values of φ_n in a recursive way from the difference equation Eq. (6.15) and verified using the closed solution provided, i. e.,

$$\begin{aligned}
\varphi_2 &= 5\varphi_1 + 6\varphi_0 + 1 = 55 - 18 + 1 = 74 = \tfrac{1}{7}[3(217) - 4(35) + 6 + 1],\\
\varphi_3 &= 5\varphi_2 + 6\varphi_1 + 1 = 370 + 66 + 1 = 437 = \tfrac{1}{7}[3(1295) - 4(217) + 42 + 0],\\
\varphi_4 &= 5\varphi_3 + 6\varphi_2 + 1 = 2185 + 444 + 1 = 2630 = \tfrac{1}{7}[3(7777) - 4(1295) + 258 + 1], \ldots
\end{aligned}$$

Example 6.4 Find the solution to the 2nd order difference equation IVP,

$$\boxed{\mathsf{F}_{n+2} = \mathsf{F}_{n+1} + \mathsf{F}_n \quad \text{with} \quad \mathsf{F}_0 = \alpha = 0\ ;\ \mathsf{F}_1 = \beta = 1} \tag{6.16}$$

Rearranging the terms in Eq. (6.16), we see that it corresponds to an homogeneous equation since the right term equals zero. The coefficients are $a_2 = 1 \neq 0$, $a_1 = -1$, and $a_0 = -1 \neq 0$. Hence, $p(z) = z^2 - z - 1$ and $D = a_1^2 - 4a_2a_0 = (-1)^2 + 4 = 5 > 0$ (positive discriminant), so Case I applies. The two roots have values,

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

After substitution and using the initial conditions, $\alpha = 0$ and $\beta = 1$, in Eq. (6.6), the solution can be expressed as follows:

$$\begin{aligned}
\mathsf{F}_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]u_{n-1}\\
&= \frac{1}{\sqrt{5}}(\phi^n - \phi_*^n)u_{n-1}\ ;\ \phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \phi_* = \frac{1-\sqrt{5}}{2}
\end{aligned}$$

This IVP defines *recursively* the *Fibonacci sequence*, its solution gives us a *closed formula* for these numbers. The ϕ number corresponds historically to the "division in extreme and mean ratio" (Euclid), the "divine proportion" (Pacioli), or the "golden section" (Ohm). As mentioned before, a closed formula refers to an expression that allows us to find directly a specific number. Thus, given n, F_n is found in a *non-recursive* way. Although outside the scope of the present exposition, we remark that there are many interesting relations between the ϕ number and its associated ϕ_* number, also denoted by the symbol $\hat{\phi}$. The most basic one is that $\phi_* = \hat{\phi} = -\phi^{-1}$. Using Eq. (6.16), the first 16 Fibonacci numbers are shown next:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Example 6.5 Find the solution to the 2nd order difference equation IVP,

$$\boxed{\varphi_{n+2} + 6\varphi_{n+1} + 18\,\varphi_n = 0 \quad \text{with} \quad \varphi_0 = \alpha = 0\ ;\ \varphi_1 = \beta = 1.} \tag{6.17}$$

The expression given in Eq. (6.17) is homogeneous with coefficients $a_2 = 1$, $a_1 = 6$, and $a_0 = 18$. Thus, $p(z) = z^2 + 6z + 18$ and $D = a_1^2 - 4a_2a_0 = 36 - 4(18) = 36 - 72 < 0$ (negative discriminant), so Case III applies. The conjugate complex roots are

$$\lambda_1 = \frac{-6 + \sqrt{-36}}{2} = -3 + 3i \quad \text{and} \quad \lambda_2 = \frac{-6 - \sqrt{-36}}{2} = - - 3 - 3i.$$

Therefore, since $\xi = -3$ and $\eta = \pm 3$, we have that $\rho = \sqrt{(-3)^2 + (\pm 3)^2} = \sqrt{18} = 3\sqrt{2}$, and $\theta = \tan^{-1}(-3/3) = \tan^{-1}(-1) = 3\pi/4$. In addition, $\varphi_0 = \alpha = 0$ and $\varphi_1 = \beta = 1$. Finally, substituting the previous values in Eq. (6.11), we obtain as solution the following sequence,

$$\boxed{\varphi_n = \frac{1}{3}(3\sqrt{2})^n \sin\frac{3n\pi}{4} u_n}$$

Example 6.6 Solve the 2nd order difference equation IVP,

$$\boxed{4\varphi_{n+2} + 25\,\varphi_n = 0 \quad \text{with} \quad \varphi_0 = \alpha = 5/2\ ;\ \varphi_1 = \beta = 1.} \tag{6.18}$$

Again, Eq. (6.18) is homogeneous with coefficients $a_2 = 4$, $a_1 = 0$, and $a_0 = 25$. Thus, $p(z) = 4z^2 + 25$ and $D = a_1^2 - 4a_2a_0 = 0 - 4(4)(25) = -400 < 0$

(negative discriminant), so Case III applies. The conjugate complex roots are given by,

$$\lambda_1 = \frac{-0+\sqrt{-400}}{2(4)} = \frac{20}{8}i = \frac{5}{2}i \quad \text{and} \quad \lambda_2 = \frac{-0-\sqrt{-400}}{2(4)} = \frac{20}{8}i = -\frac{5}{2}i.$$

Therefore, since $\xi = 0$ and $\eta = \pm 5/2$, we have that $\rho = \sqrt{0^2 + (\pm 5/2)^2} = 5/2$, and $\theta = \tan^{-1}((5/2)/0) = \tan^{-1}(\infty) = \pi/2$. In addition, $\varphi_0 = \alpha = 2/5$ and $\varphi_1 = \beta = 1$. Hence, substituting these numbers in Eq. (6.11), the solution is given by the sequence,

$$\boxed{\varphi_n = \left(\frac{5}{2}\right)^n \left(\frac{5}{2}\cos\frac{n\pi}{2} + \frac{2}{5}\sin\frac{n\pi}{2}\right)u_n}$$

Example 6.7 Our last exercise will find a sequence that solves the IVP,

$$\boxed{\mathfrak{F}_{n+2} = \mathfrak{F}_{n+1} + \mathfrak{F}_n + \mathsf{F}_n \quad \text{with} \quad \mathfrak{F}_0 = \alpha = 0\ ;\ \mathfrak{F}_1 = \beta = 1} \tag{6.19}$$

The stated IVP defines what is known as the *Fibonacci sequence of 2nd order*; being F_n the n-th Fibonacci number, then the task is to determine a closed expression in terms of F_n for $\mathfrak{F}_n$. Rewriting Eq. (6.19), one can see, as displayed next, that the second order difference equation is heterogeneous because $\mathsf{F}_n \neq 0$ for all $n \geq 0$,

$$\mathfrak{F}_{n+2} - \mathfrak{F}_{n+1} - \mathfrak{F}_n = \mathsf{F}_n\,. \tag{6.20}$$

For the given initial values, $\mathfrak{F}_0 = 0$ and $\mathfrak{F}_1 = 1$, the solution is given by

$$\begin{aligned}\mathfrak{F}_n = \mathfrak{F}_n^c + \mathfrak{F}_n^p &= \mathsf{F}_n + \sum_{i=1}^{n-1} \mathsf{F}_i \mathsf{F}_{n-1-i} \\ &= \mathsf{F}_n + \sum_{i=1}^{m} \mathsf{F}_i \mathsf{F}_{m-i} \quad \text{where} \quad m = n-1\,.\end{aligned}$$

Since,

$$\mathsf{F}_i = \frac{1}{\sqrt{5}}(\phi^i - \phi_*^i)u_{i-1} \quad \text{and} \quad \mathsf{F}_{m-i} = \frac{1}{\sqrt{5}}(\phi^{m-i} - \phi_*^{m-i})u_{m-i-1}\,,$$

then,

$$\sum_{i=1}^{m} \mathsf{F}_i \mathsf{F}_{m-i} = \frac{1}{5}\sum_{i=1}^{m}(\phi^i - \phi_*^i)(\phi^{m-i} - \phi_*^{m-i})u_{i-1}u_{m-i-1}$$
$$= \frac{1}{5}\Big[\sum_{i=1}^{m-1}(\phi^i - \phi_*^i)(\phi^{m-i} - \phi_*^{m-i})\Big]u_{m-2}\,. \tag{6.21}$$

In the second equality of Eq. (6.21) we can see that,

$$u_{m-1-i} = \left\{\begin{array}{lll} 1 & \text{if} & m-1 \geqslant i \\ 0 & \text{if} & m-1 < i \end{array}\right\} \quad \text{and} \quad u_{i-1}\Big|_{i=m-1} = u_{m-2}\,.$$

The partial products between monomials within the summation symbol (shown left) and their respective sums (shown right) are given next,

$$\begin{aligned} +\phi^i\phi^{m-i} &= \phi^m \quad ; \quad (m-1)\phi^m\,, \\ -\phi^i\phi_*^{m-i} &= \phi_*^m\left(\frac{\phi}{\phi_*}\right)^i \quad ; \quad -\phi_*^m\left(\frac{\phi}{\phi_*}\right)\left[\frac{(\phi/\phi_*^{m-1}-1}{(\phi/\phi_*)-1}\right] = \frac{\phi_*^m\phi - \phi^m\phi_*}{\phi-\phi_*}\,, \\ -\phi_*^i\phi^{m-i} &= \phi^m\left(\frac{\phi_*}{\phi}\right)^i \quad ; \quad -\phi^m\left(\frac{\phi_*}{\phi}\right)\left[\frac{(\phi_*/\phi)^{m-1}-1}{(\phi_*)\phi)-1}\right] = \frac{\phi_*^m\phi - \phi^m\phi_*}{\phi-\phi_*}\,, \\ +\phi_*^i\phi_*^{m-i} &= \phi_*^m \quad ; \quad (m-1)\phi_*^m. \end{aligned}$$

Using the identities, $\phi^m = \mathsf{F}_m\phi + \mathsf{F}_{m-1}$ and $\phi_*^m = \mathsf{F}_m\phi_* + \mathsf{F}_{m-1}$, it follows that, $(m-1)\,(\phi^m + \phi_*^m) = (m-1)\mathsf{F}_m(\phi+\phi_*) + 2(m-1)\mathsf{F}_{m-1}$. Reintroducing the numerical factor $1/5$ we get

$$\frac{(m-1)}{5}(\phi^m + \phi_*^m) = \frac{(m-1)}{5}\mathsf{F}_m + \frac{2m}{5}\mathsf{F}_{m-1} - \frac{2}{5}\mathsf{F}_{m-1}\,. \tag{6.22}$$

Similarly,

$$\frac{2}{5}\left(\frac{\phi_*^m\phi - \phi^m\phi_*}{\phi-\phi_*}\right) = \frac{2}{5\sqrt{5}}(\phi_*^m\phi - \phi^m\phi_*)$$
$$= \frac{2}{5\sqrt{5}}(\mathsf{F}_m\phi_*\phi + \mathsf{F}_{m-1}\phi - \mathsf{F}_m\phi\phi_* - \mathsf{F}_{m-1}\phi_*)$$
$$= \frac{2}{5\sqrt{5}}\mathsf{F}_{m-1}(\phi-\phi_*) = \frac{2}{5}\mathsf{F}_{m-1}\,. \tag{6.23}$$

Therefore, adding Eqs. (6.22) and (6.23), the particular solution is given by,

$$\mathfrak{F}_n^p = \frac{(m-1)}{5}\mathsf{F}_m + \frac{2m}{5}\mathsf{F}_{m-1} - \frac{2}{5}\mathsf{F}_{m-1} + \frac{2}{5}\mathsf{F}_{m-1}\,,$$

or, recalling that $m = n-1$, the previous expression look like,

$$\mathfrak{F}_n^p = \frac{1}{5}[(n-2)\mathsf{F}_{n-1} + 2(n-1)\mathsf{F}_{n-2}]u_{n-3}\,.$$

Thus, the *unique solution* to the proposed IVP is:

$$\boxed{\mathfrak{F}_n = \mathsf{F}_n + \frac{1}{5}[(n-2)\mathsf{F}_{n-1} + 2(n-1)\mathsf{F}_{n-2}]u_{n-3}} \tag{6.24}$$

If $n = 0, 1, 2$, $\mathfrak{F}_n = \mathsf{F}_n$. Otherwise, if $n \geqslant 3$, we use the equalities, $\mathsf{F}_{n-1} = \mathsf{F}_{n+1} - \mathsf{F}_n$ and $\mathsf{F}_{n-2} = \mathsf{F}_n - \mathsf{F}_{n-1} = \mathsf{F}_n - \mathsf{F}_{n+1} + \mathsf{F}_n = 2\mathsf{F}_n - \mathsf{F}_{n+1}$, to rewrite $\mathfrak{F}_n$. Hence,

$$\mathfrak{F}_n = \mathsf{F}_n + \frac{1}{5}(n\mathsf{F}_{n+1} - n\mathsf{F}_n - 2\mathsf{F}_{n+1} + 2\mathsf{F}_n) + \frac{2}{5}(2n\mathsf{F}_n - n\mathsf{F}_{n+1} - 2\mathsf{F}_n + \mathsf{F}_{n+1})\,,$$

or equivalently,

$$\boxed{\mathfrak{F}_n = \frac{1}{5}[3(n+1)\mathsf{F}_n - n\mathsf{F}_{n+1}]\,.} \tag{6.25}$$

It is not difficult to verify that the last expression is true $\forall n \geqslant 0$. Based on Eq. (6.19), here are the first 16 Fibonacci numbers of second order:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathfrak{F}_n$	0	1	1	3	5	10	18	33	59	105	185	324	564	977	1685	2895

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