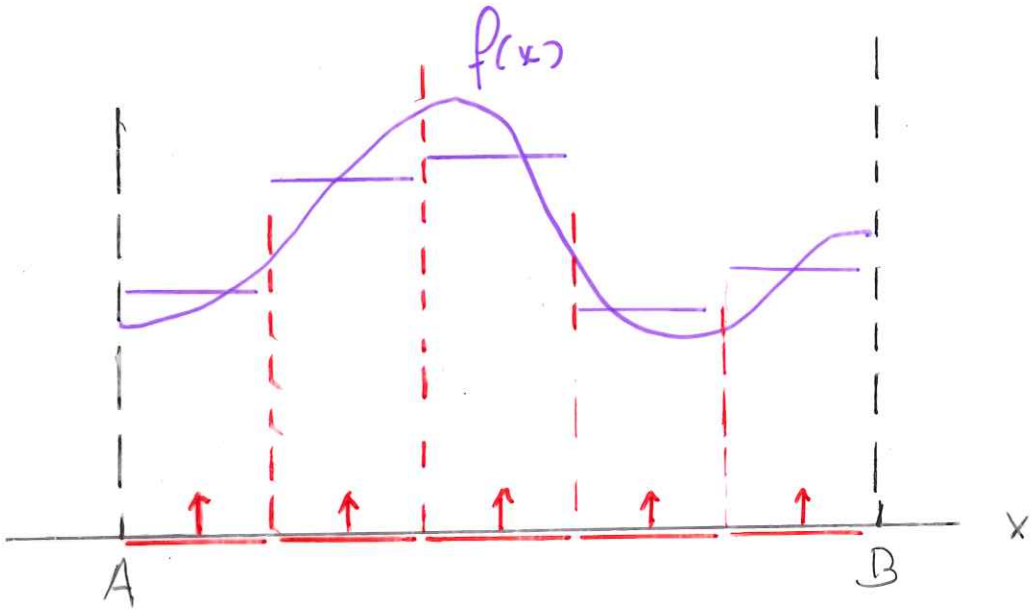
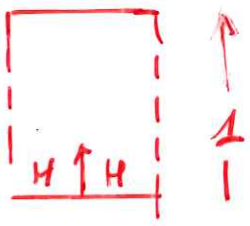
13

INVERSIONES NUMÉRICAS

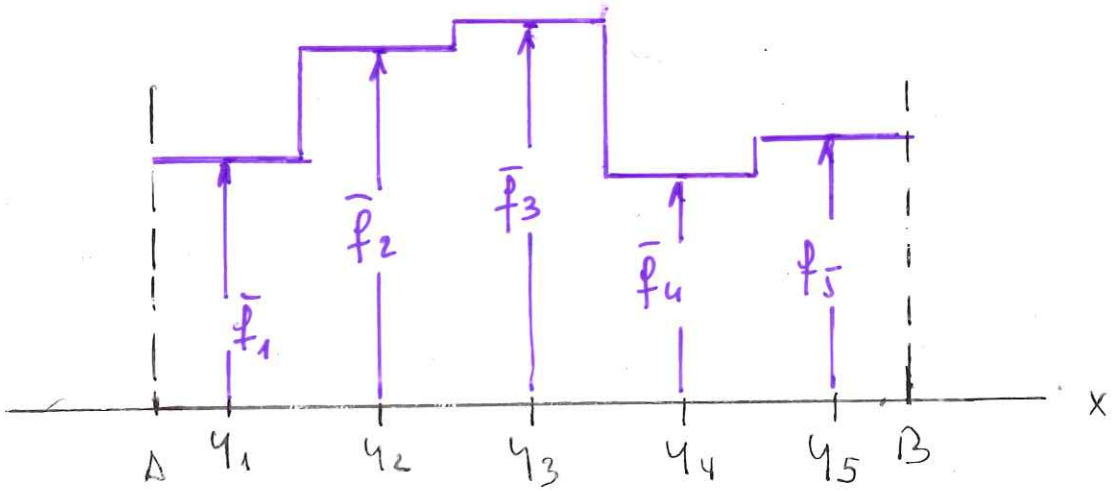
13.1

INVERSIÓN NUMÉRICA "NAIF"





DETECTOR IDEAL, CON ESA FORMA



$$y_1 = A + H$$

$$y_2 = y_1 + 2H$$

$$y_3 = y_2 + 2H$$

$$y_4 = y_3 + 2H$$

$$y_5 = y_4 + 2H$$

$$y_5 + H = B$$

$$g_1 \equiv g(y_1) = \int_{y_1 - H}^{y_1 + H} f(x)\,dx = 2H\ \bar{f}_1$$

$$g_2 \equiv g(y_2) = \int_{y_2 - H}^{y_2 + H} f(x)\,dx = 2H\ \bar{f}_2$$

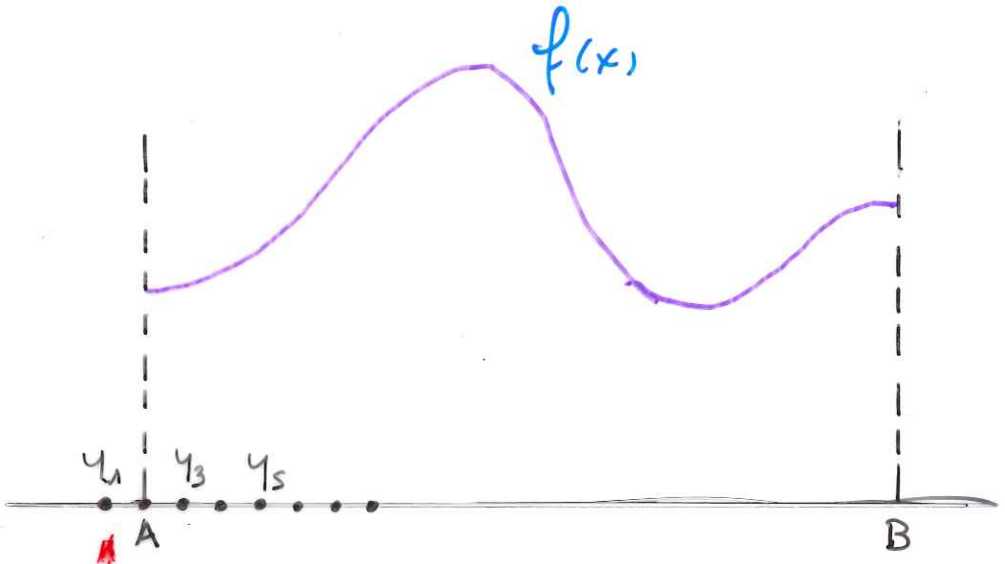
etc $\Rightarrow$ $\bar{f}_1,\ \bar{f}_2,\ \bar{f}_3\ \bar{f}_4,\ \bar{f}_5$

EN CIERTO MODO VALORES MEDIOS

$$g(y) = \int_A^B K(y, x)\ f(x)\ dx$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{pmatrix} = 2H \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \\ \bar{f}_4 \\ \bar{f}_5 \end{pmatrix}$$

¿Se puede deducir algo más que los primeros valores medios?



1
2
3
4
5
6

$\delta < 2H$

Aquí $\delta = \frac{H}{2}$

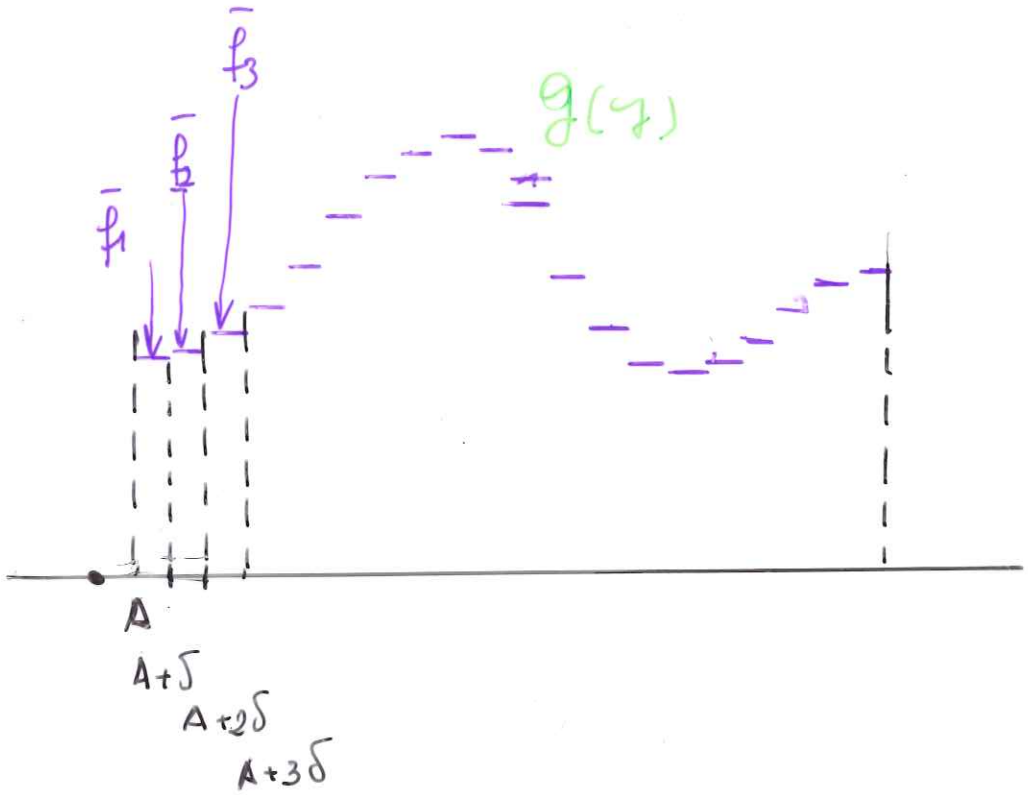
$y_1 = A - H + \delta = A - \delta$

$y_2 = y_1 + \delta = A - H + 2\delta = A$

$y_3 = y_2 + \delta = A - H + 3\delta = A + \delta$

$y_4 = y_3 + \delta = A - H + 2\delta = A + 2\delta$

etc.



se mide

$$g_1 \equiv g(\gamma_1) = \int_A^{A+\delta} f(x)\,dx = \delta\,\bar{f}_1$$

$$g_2 \equiv g(\gamma_2) = \int_A^{A+2\delta} f(x)\,dx = g_1 + \delta\,\bar{f}_2 = \delta(\bar{f}_1 + \bar{f}_2)$$

$$g_3 \equiv g(\gamma_3) = \int_A^{A+3\delta} f(x)\,dx = g_2 + \delta\,\bar{f}_3 = \delta(\bar{f}_1 + \bar{f}_2 + \bar{f}_3)$$

$$g_4 \equiv g(\gamma_4) = \int_A^{A+4\delta} f(x)\,dx = g_3 + \delta\,\bar{f}_4 = \delta(\bar{f}_1 + \bar{f}_2 + \bar{f}_3 + \bar{f}_4)$$

$$g_5 \equiv g(\eta_5) = \int_{\eta_5 - H}^{\eta_5 + H} f(x)\,dx = -g_1 + g_4 + \delta \bar{f}_5$$

$$= \delta(\bar{f}_1 + \bar{f}_3 + \bar{f}_4 + \bar{f}_5)$$

$$g_6 = g(\eta_6) = \int_{\eta_6 - H}^{\eta_6 + H} f(x)\,dx = -(g_2 - g_1) + g_5 + \delta \bar{f}_6$$

$$= \delta(\bar{f}_3 + \bar{f}_4 + \bar{f}_5 + \bar{f}_6$$

$$g_7 = -(g_3 - g_2) + g_6 + \delta \bar{f}_7$$

$$= \delta(\bar{f}_4 + \bar{f}_5 + \bar{f}_6 + \bar{f}_7)$$

$$\begin{vmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ \vdots \end{vmatrix} = \delta \begin{vmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & \ddots & \\ 0 & 1 & 1 & 1 & \ddots & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & & \end{vmatrix} \begin{vmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \\ \bar{f}_4 \\ \bar{f}_5 \\ \bar{f}_6 \\ \bar{f}_7 \\ \vdots \end{vmatrix}$$

INVERSA

$$\bar{f}_1 = \frac{1}{\delta} g_1$$

$$\bar{f}_2 = \frac{1}{\delta}(g_2 - g_1)$$

$$\bar{f}_3 = \frac{1}{\delta}(g_3 - g_2)$$

$$\bar{f}_4 = \frac{1}{\delta}(g_4 - g_3)$$

$$\bar{f}_5 = \frac{1}{\delta}(g_5 - g_4 + g_1)$$

$$\bar{f}_6 = \frac{1}{\delta}(g_6 - g_5 + g_2 - g_1)$$

$$\bar{f}_7 = \frac{1}{\delta}(g_7 - g_6 + g_3 - g_2)$$

$$\bar{f}_8 = \frac{1}{\delta}(g_8 - g_7 + g_4 - g_3)$$

etc

PROBLEMAS:

$g_k \to \bar{g}_k (1+\varepsilon_k)$ errores

$$\bar{f}_1 = \frac{1}{\delta} \bar{g}_1 (1+\varepsilon_1)$$

$$\bar{f}_2 = \frac{1}{\delta} \left[\bar{g}_2 (1+\varepsilon_2) - \bar{g}_1 (1+\varepsilon_1) \right] =$$

$$= \frac{1}{\delta} \left[\bar{g}_2 - \bar{g}_1 + (\bar{g}_2 \varepsilon_2 - \bar{g}_1 \varepsilon_1) \right] =$$

$$= \frac{1}{\delta} (\bar{g}_2 - \bar{g}_1) \left[1 + \frac{\bar{g}_2 \varepsilon_2 - \bar{g}_1 \varepsilon_1}{\bar{g}_2 - \bar{g}_1} \right]$$

puede haber problemas si $g_2 \simeq g_1$ etc

13.2

INVERSIÓN MATRICIAL

DISCUSIÓN SOBRE LA CONSTRUCCIÓN DEL NÚCLEO VIA INTEGRACIÓN NUMÉRICA DIRECTA

DISCRETIZACIÓN MATRICIAL DIRECTA

$$\int_a^b K(x, y)\, f(x)\, dx = g(y)$$

PARA CADA $y =$

$$\int_a^b K(x, y_k)\, f(x)\, dx = g(y_k)$$

ADMITIENDO UNA FORMULA DE INTEGRACIÓN NUMÉRICA PARA LA INTEGRAL SOBRE

$$\sum_j W_j\, K(x_j, y_k)\, f(x_j) = g(y_k)$$

TOMANDO TANTOS VALORES DE y_k Y DE $g(y_k)$ COMO SE HAN ELEGIDO PARA LA INTEGRAL SOBRE x Y EN TÉRMINOS DE ORDENADAS DISCRETAS x_j (UN NUMERO MAYOR PARA y_k PUDIERA SER MÁS CONVENIENTE AUN -MINIMOS CUADRADOS- PERO CUIDADO NO REPETIR ECUACIONES)

TENDREMOS UN SISTEMA LINEAL DE CUYA INVERSION SE DEDUCIRÁ EL CONJUNTO DE VALORES $f(x_j)$

Inverse Problems in Astronomy

a guide to inversion strategies for remotely sensed data

Ian J D Craig
BSc, PhD

Senior Lecturer in Mathematics
University of Waikato, Hamilton, New Zealand

John C Brown
BSc, PhD, DSc, FRAS, FRSE

Professor of Astrophysics
University of Glasgow

Adam Hilger Ltd, Bristol and Boston

5.2 Numerical Inversion Methods

To illustrate the basic methods involved it is sufficient to consider the reduction of the Fredholm integral equation of the first kind

$$\int_a^b k(x, y)f(y)\,dy = g(x) \qquad a \leqslant x \leqslant b \tag{5.1}$$

to matrix form. Equations of the Volterra type can generally be accommodated within the same basic framework (see e.g. §5.4).

It is instructive, first of all, to comment on the effect of specifying the data function only at discrete points, x_i say, where $a \leqslant x_1 < \ldots < x_i < \ldots < x_m \leqslant b$. Equation (5.1) then becomes

$$\int_a^b k(x_i, y)f(y)\,dy = g(x_i) \qquad i = 1, 2, \ldots, m. \tag{5.2}$$

Although no error is involved in this discretisation it is clear that many functions $f(y)$ can satisfy equation (5.2) without being solutions of equation (5.1). This reflects the fact that the discretised data points can never uniquely represent the data function, in that, any polynomial of degree $(m-1)$ or greater can pass through the node points of g. Thus a strong element of information loss has already crept into the functional problem by its discretisation over the x variable.

It follows that the practical inversion problem can never possess an unambiguous solution, even in the absence of data noise.

5.2.1 *Matrix–quadrature methods*

This is probably the simplest of all numerical methods. The integral in equation (5.2) is approximated directly by a quadrature sum so that

$$\sum_{j=1}^{n} k(x_i, y_j)\, w_j f(y_j) \simeq g(x_i) \qquad i = 1, 2, \ldots, m \tag{5.3}$$

where the w_j are weighting coefficients associated with the quadrature rule. A first-order approximation is obtained in the simplest case of uniform weighting ($w_j = \Delta y = (b-a)/(n-1)$) but more accuracy can, in principle, be derived by employing the trapezoidal rule or the Simpson quadrature (see Chapter 6 for detailed application). These correspond respectively, to piecewise constant, linear and quadratic approximations to the integrand in equation (5.2). Still greater accuracy in representation (5.3) can be derived by employing more sophisticated quadrature formulae, for example those derived by Gaussian integration techniques. In all cases, however, an $m \times n$ matrix system is derived, i.e.

$$\sum_{j=1}^{n} A_{ij} f_j = g_i \qquad i = 1, 2, \ldots, m \tag{5.4}$$

The central problem is then to determine the differential emission measure function $\xi(T)$, describing the distribution of material over temperature in the source, from the bremsstrahlung spectrum $\varphi(\varepsilon_i)$. The 'integral' equation to be inverted (cf equation (2.32)) takes the discrete form

$$\varphi(\varepsilon_i) = \int_T k(\varepsilon_i, T)\, \xi(T)\, \mathrm{d}T \tag{7.9}$$

for each ε_i at which φ is measured, with the kernel

$$k(\varepsilon, T) = \frac{A}{T^{1/2}} \exp(-\varepsilon/T) \tag{7.10}$$

where A is a scaling factor and T is in energy units.

In practice it is often convenient to express equation (7.9) as a standard Laplace transform (cf equation (2.33))

$$\varphi(\varepsilon_i) = \int_0^\infty \exp(-\varepsilon_i t)\, G(t)\, \mathrm{d}t. \tag{7.11}$$

7.3.2 *Classical inversions*

(a) Matrix–quadrature. We begin by exploring the following problem. Suppose we are given n noise free measurements of the power law spectrum equation (7.13) in some typical energy range, say 3–10 keV. Can we obtain a satisfactory n-point approximation to the true solution, equation (7.14) by invoking the classical quadrature technique? (In what follows we discretise equation (7.9) using the Simpson quadrature, generally taking constant logarithmic intervals in the $\varepsilon - T$ variables (see §7.4.2).)

First we observe from figure 7.3 that the linear system obtained by discretising equation (7.9) is generally extremely badly conditioned, even for relatively low order numerical approximations ($n = 5$–9 say). Curve B for example, illustrates the behaviour of the condition number C_A for the energy band 3–10 keV and the temperature span 5–60×10^6 K. Of special interest is the rapid (exponential) rise in C_A as the number of quadrature points is increased. We see by equation (5.27) that to guarantee a stable solution in only a nine point approximation requires relative data errors approaching 1 part in 10^8, obviously a quite impossible requirement. Nor is the matter radically improved by choosing different wavelength and temperature bands (curves A and C): the rapid rise in C_A as the discretisation mesh is 'filled in' is a universal trend that is not much affected by the temperature–photon energy representation.

that the data function namely

$$\varphi(s) = \int_0^\infty \exp(-st)N(t)\,dt \tag{7.40}$$

is given by the Laplace transform of the source function. In constructing numerical solutions we deal only with discretised approximations to equation (7.40).

7.4.2 *Numerical inversion method*

We employ two methods to reduce equation (7.40) to matrix form

$$\varphi_i = \sum_{j=1}^{n} K_{ij}N_j \qquad i = 1, \ldots, m. \tag{7.41}$$

First we invoke the first-order product integration method introduced in §5.2.3(see §§5.3.3 (Example A) and 5.4.3 for applications). Although only exact for piecewise constant source functions, this method has the attraction of not requiring a uniformly incremented t mesh, nor any strategem for truncating the infinite t integration interval. The kernel matrix is given by

$$K_{ij} = \frac{1}{s_i}\left[\exp(-s_i t_{j-1}) - \exp(-s_i t_j)\right] \tag{7.42}$$

where an individual partition $(t_j - t_{j-1})$ can be chosen in *any* convenient manner.

Of course greater accuracy can be derived by employing a conventional matrix quadrature, for example Simpson's rule. In this case the fundamental t interval must be partitioned using equal increments with a consequent loss of flexibility in the representation of the source function. Moreover, in practice, the large variation over both frequency (ν) and energy (γ) makes it expedient to introduce logarithmic variables (as in §7.3.1)

$$x = \ln(s) \qquad y = \ln(t) \tag{7.43}$$

so that the kernel in equation (7.40) becomes

$$k(x, y) = \exp(y - e^{x+y}) \tag{7.44}$$

where $-\infty < y < \infty$. Assuming an appropriate truncation of this interval (see below) the kernel is discretised by taking a uniform increment Δy whence

$$K_{ij} = w_j k(x_i, y_i)\,\Delta y \tag{7.45}$$

w_j being the weights associated with Simpson's rule namely $(1, 2, 4, 2, \ldots, 2, 4, 2, 1)$ with n odd. The inversion problem is then reduced to equation (7.41) on identifying $\varphi_i \equiv \exp x_i$ and $N_j \equiv N\exp y_j$. This system is *formally* far more accurate than equation (7.42), being exact for cubic rather than zeroth-order polynomial source functions.

Two strategies will be adopted to stabilise the matrix inversion problem.

THE NUMERICAL TREATMENT OF INTEGRAL EQUATIONS

By

CHRISTOPHER T. H. BAKER M.A., D.Phil.

Reader in Mathematics,
University of Manchester

153.107

CLARENDON PRESS · OXFORD
1977

5.16. *Quadrature methods and expansion methods*

Let us investigate briefly how a quadrature method may be set up for an equation of the first kind. We shall then indicate the *potentially* unsatisfactory nature of the method. Modifications of the method, intended to alleviate some of its shortcomings, will be discussed later.

From eqn (5.104) we have, on replacing the integral by a rule with weights w_j and abscissae y_j $(j = 0,1,2,\ldots,n)$, the approximation

$$\sum_{j=0}^{n} w_j K(x,y_j) f(y_j) \simeq g(x) .$$

If we write

$$\sum_{j=0}^{n} w_j K(x,y_j) \tilde{f}(y_j) = g(x) \qquad (5.107)$$

and seek values of $\tilde{f}(y_j)$ $(j = 0,1,2,\ldots,n)$ satisfying (5.107), we shall not in general find a solution. A solution exists only if $g(x)$ is a linear combination of the functions $K(x,y_j)$ $(j = 0,1,2,\ldots,n)$ and, even if the integral equation has a solution, this is not guaranteed. However, if we select $z_0,z_1,\ldots,z_m$ we may be able to satisfy the equations

$$\sum_{j=0}^{n} w_j K(z_i,y_j) \tilde{f}(y_j) = g(z_i) \quad (i = 0,1,2,\ldots,m) . \qquad (5.108)$$

We shall suppose that $m = n$, until further notice, and these equations then define $\tilde{f}(y_0),\ldots,\tilde{f}(y_n)$ provided $\det[K(z_i,y_j)] \neq 0$ (in particular, if the functions $K(x,y_j)$ form a Haar set). There is no necessity in these equations to choose $z_i = y_i$, though it may be thought natural to do so. We shall demonstrate by examples that eqns (5.108) do *not* lead to a generally reliable method of approximating the values of a solution of the integral equation. We shall proceed under the general assumption that $K(x,y)$ and $g(x)$ are continuous.

Encuentra problemas pero no sabe muy bien la razón.

J. ASSOC. COMPUTING MACHINERY
9, 84, 97, 1962

A Technique for the Numerical Solution of Certain Integral Equations of the First Kind*

DAVID L. PHILLIPS†

Argonne National Laboratory, Argonne, Illinois

Introduction

The general linear equation may be written as

$$h(x)f(x) + \int_a^b K(x, y)f(y)\, dy = g(x) \qquad (a \leqq x \leqq b)$$

where the known functions $h(x)$, $K(x, y)$ and $g(x)$ are assumed to be bounded and usually to be continuous. If $h(x) \equiv 0$ the equation is of *first kind*; if $h(x) \neq 0$ for $a \leqq x \leqq b$, the equation is of *second kind*; if $h(x)$ vanishes somewhere but not identically, the equation is of *third kind*. If the range of integration is infinite or if the kernel $K(x, y)$ is not bounded, the equation is *singular*. Here we will consider only nonsingular linear integral equations of the first kind:

$$\int_a^b K(x, y)f(y)\, dy = g(x) \qquad (a \leqq x \leqq b) \qquad (1)$$

There is extensive literature on equations of the second kind, but literature on linear equations of the first kind is sparse. However, several methods for solving equations of the first kind numerically have been proposed [1–10]. No method has been very successful for arbitrary kernels when the function $g(x)$ is known with only modest accuracy. The reason for this is inherent in the equation itself. Think of the equation as a linear operator, operating on $f(y)$ to produce $g(x)$. This operator does not have a bounded inverse (it may not even have an inverse, but we will assume here that it does) which can be seen as follows. Let $f(y)$ be the solution to (1) and add to it the function $f_m = \sin my$. For any integrable kernel it is known that $g_m \equiv \int_a^b K(x, y) \sin(my)\, dy \to 0$ as $m \to \infty$. Hence only an infinitesimal change g_m in g causes a finite change f_m in f (i.e. the equation is unstable). Also, one would expect that $g_m \to 0$ as $m \to \infty$ faster for flat smooth kernels than for sharply peaked kernels (indeed if $K(x, y)$ were the δ-function, $K(x, y) = \delta(x - y)$, then $g_m = f_m$ would not approach zero). Hence we conclude that the success in solving equation (1) by any method depends to a large extent on the accuracy of $g(x)$ and the shape of $K(x, y)$.

* Received June, 1961.
† Based on work performed under the auspices of the U. S. Atomic Energy Commission.

From equations (13), the matrix method described here merely replaces A by $A + \gamma B$ where B is a certain matrix whose elements depend only on A and γ is an arbitrary non-negative parameter which controls the amount of smoothing. Increasing γ produces greater smoothing. It follows from the second equation in (13) and (5) that e is approximately proportional to γ so that only a very few values of γ need to be used in order to find one giving e the desired magnitude. However, one additional matrix inversion is needed for each new value of γ used. The value of e is determined from the accuracy of the g_j.

Examples

Example 1. Let the problem be the following:

$$\int_{-3}^{3} K(x - \lambda) f(x)\, dx = g(\lambda)$$

where

$$K(z) = 1 + \cos\frac{\pi z}{3}, \qquad |z| \leqq 3,$$

$$= 0, \qquad |z| > 3;$$

$$g(z) = (6 + \lambda)\left(1 - \frac{1}{2}\cos\frac{\pi\lambda}{3}\right) - \frac{9}{2\pi}\sin\frac{\pi\lambda}{3}, \qquad |z| \leqq 6,$$

$$= 0, \qquad |z| \geqq 6.$$

The solution to this problem is $f(x) = K(x)$. Hence we can easily check the numerical solution against the true solution. Let us first take $n = 12$ (13 points) and use Simpson's rule for the quadrature formula. The truncation error is about .4. The values of g are rounded off so that the maximum error in $g(\lambda_j)$ is .00005. Table I and Figure 1 give the comparison between the true solution and the numerical solution for several choices of γ. Since the solution is symmetric

TABLE I

λ	true values of f	$\gamma = 0$	$\gamma = .0011$	$\gamma = .011$	$\gamma = .03$	$\gamma = .1$	$\gamma = .5$	$\gamma = 1$
0	2.0000	2.943	2.356	2.082	2.017	1.948	1.843	1.796
1	1.5000	1.154	1.404	1.478	1.488	1.489	1.476	1.463
2	.5000	.694	.388	.464	.503	.559	.651	.686
3	.0000	.014	.077	−.002	−.016	−.014	.016	.042
4	.0000	.000	−.034	.031	.019	−.026	−.105	−.120
5	.0000	.000	.005	.029	.038	.034	.009	−.009
6	.0000	.000	−.012	−.128	−.133	−.052	.110	.131
ave. $\lvert\epsilon_j\rvert$		0	.010	.020	.026	.047	.105	.140
max. $\lvert\epsilon_j\rvert$		0	.016	.035	.053	.096	.203	.242

CONCLUSION RAPIDA

where $g_i = g(x_i)$, $f_j = f(x_j)$ and $A_{ij} = w_j k_{ij}$. It is a simple matter, in principle, to resolve this system (for $m \geqslant n$) to yield an n-point numerical approximation to the solution. In particular, taking $n = m$ yields an 'exact' numerical solution—one that 'models' the discrete data function to within numerical round-off.

At first sight it would appear that the system in equation (5.4) provides a simple, robust numerical inversion technique for resolving equation (5.2). In practice however, the quadrature method runs up against severe difficulties. In particular, when an attempt is made to refine the discretised solution by increasing the number of support points in the source function representation, the conditioning (cf §5.3.2) of the matrix system generally worsens disastrously, leading to a severe magnification of data (and discretisation) errors and eventually to quite meaningless, highly oscillatory, numerical solutions. For the simplest case of a square matrix system ($m = n$) the numerical inversion deteriorates as the matrix operator $\mathbf{A}^{-1}$ begins to reflect, more and more closely, the unboundedness of the original integral operator $\mathcal{K}^{-1}$ (cf §4.3.1). The problem is generally most severe for strongly smoothing kernels (cf §5.3) but, in any given case, an optimum representaton can usually be found by judiciously selecting the spacing of the discretisation mesh (x_i, y_j) in addition to the number of support points (n) in the numerical solution. It is disturbing, none the less, to obtain a numerical formulation of the problem, the solution of which depends critically on what appears to be an essentially arbitrary numerical mesh. It also conflicts with the notions of classical numerical analysis to the extent that numerical solutions are expected to show *some degree of convergence* as discretisation errors are systematically reduced—provided of course that round-off error is kept at bay.

We return to a more quantitative discussion of the numerical instability in §5.3.

1ª CONCLUSIÓN

UTILIZANDO UN MÉTODO DE INTEGRACIÓN NUMÉRICA DIRECTA PARA REPRESENTAR NUMÉRICAMENTE UNA TRANSFORMADA INTEGRAL HAY, GENERALMENTE, PROBLEMAS (MUCHOS Y MUY GRAVES) (BAER 2.)

PERO, SOBRE TODO CUANDO SE UTILIZA SIMPSON

Y, LO MAS GRAVE ES QUE LO RECOMIENDAN

Y MAS GRAVE AUN ES QUE ESTO ES Y ERA PREVISIBLE

Y QUE LOS QUE LO RECOMIENDAN GOZAN DE UN GRAN PREDICAMENTO...

JUGUEMOS A INTEGRAR POR SIMPSON

Si necesitamos evaluar numericamente la integral

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(x-\eta)^2} dx$$

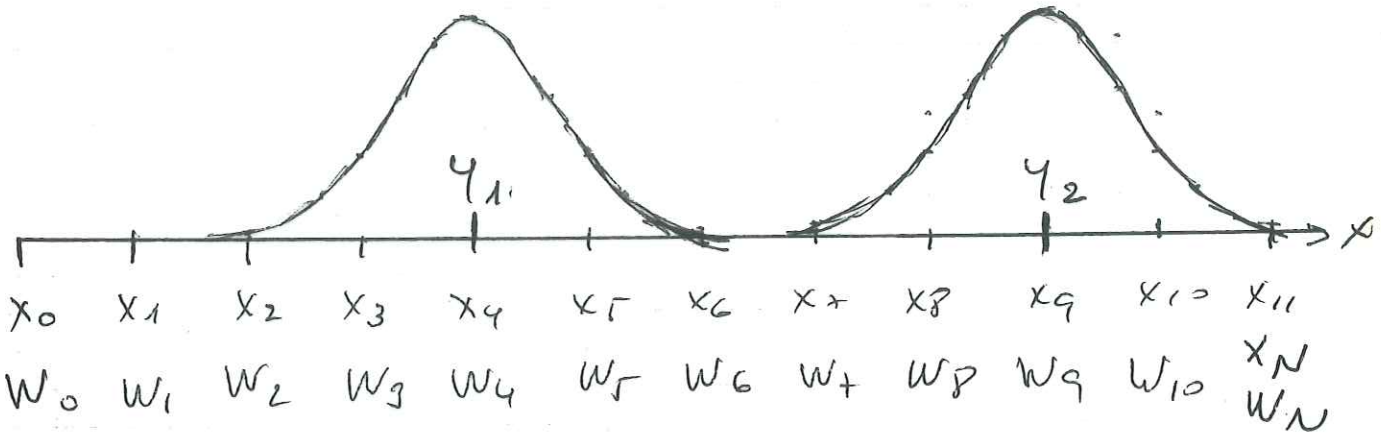
para diversos valores de η

PARA TODOS ELLOS VALE 1

en una lógica matematica deberiamos tomar una serie de ordenadas discretas x_k - y los correspondientes pesos de integración w_k :

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(x-\eta)^2} dx \simeq \sum_k w_k \frac{1}{\sqrt{\pi}} e^{-(x_k-\eta)^2}$$

tales que $x_k - \eta$ coincidieran con las abscisas de una "discretización de Hermite"



pero como se trata de evaluar la integral para diversos valores de y:

$y_1, y_2 \cdots y_j$

las abcisas x_k, referidas a cualquier y_j: $x_k - y_1 \quad x_k - y_2 \cdots x_k - y_j$ no coincidirán con los valores de Hermite (para todos y_j).

POR ELLO HABRÁ QUE PENSAR EN UNA DIVISION (DISCRETIZACION) REGULAR DE LA VARIABLE x:

$x_0, x_1, x_3 \cdots x_k$

$x_0 - x_0 = x_2 - x_1 = x_3 - x_2$ etc

Y TOMAR UNO (VARIOS) DE ESTOS PUNTOS TAMBIEN PARA LA VARIABLE Y

NUMERICAMENTE

$$\int_{-\infty}^{+\infty} \frac{e^{-(x-\gamma)^2}}{\sqrt{\pi}} dx = \sum_{k=0}^{N} W_k \frac{e^{-(x_k-\gamma)^2}}{\sqrt{\pi}}$$

TRAPECIOS

$W_0 = \frac{\Delta}{2}$ $\quad W_1 = W_2 = W_3 = \dots = W_k = \dots W_{N-1} = \Delta$

$W_N = \frac{\Delta}{2}$ $\quad \Delta$ distancia entre dos abscisas consecutivas

$\Delta = 1$

$x_k - \gamma$	$\frac{1}{\sqrt{\pi}} e^{-(x_k-\gamma)^2}$
0	$5.641896\ 10^{-1}$
1	$2.075537\ 10^{-1}$
2	$1.033349\ 10^{-2}$
3	$6.962659\ 10^{-5}$
4	$6.349117\ 10^{-8}$

Si estamos suficientemente lejos de los bordes, todos los PESOS SON IGUALES

$W_k = \Delta = 1$

$$\sum_{k=0}^{N} \frac{W_k e^{-(x-\gamma)^2}}{\sqrt{\pi}} = \frac{e^{-0^2}}{\sqrt{\pi}} + 2\frac{e^{-1^2}}{\sqrt{\pi}} + 2\frac{e^{-2^2}}{\sqrt{\pi}} + 2\frac{e^{-3^2}}{\sqrt{\pi}} + 2\frac{e^{-}}{\sqrt{\pi}}$$

$= 1.000103$

El resultado numérico es excelentemente bueno PERO ES UNA CASUALIDAD

Si $\Delta = 1/2$

$x_k - \eta$	$\frac{1}{\sqrt{\pi}} e^{-(x_k-\eta)^2}$
0	$5.641896 \cdot 10^{-1}$
0.5	$4.393693 \cdot 10^{-1}$
1	$2.075537 \cdot 10^{-1}$
1.5	$5.946514 \cdot 10^{-2}$
2	$1.039949 \cdot 10^{-2}$
2.5	$1.089142 \cdot 10^{-3}$
3	$6.562653 \cdot 10^{-5}$
3.5	$2.699713 \cdot 10^{-6}$
4	$6.349117 \cdot 10^{-8}$

$W_k = \frac{1}{\Sigma}$ vale todo salvo en los extremos

$$\sum_{k=1}^{N} W_k \frac{e^{-(x_k-\eta)^2}}{\sqrt{\pi}} = 0.999999999$$

el resultado de integrar por trapecios la gaussiana es excelente (pero es una casualidad)

SIMPSON

$W_0 = \frac{1}{3}\Delta$ $\qquad$ $W_N = \frac{1}{3}\Delta$

$W_1 = W_3 = W_5 = \ldots \quad \frac{2}{3}\Delta$

$W_2 = W_4 = W_6 = \ldots \quad \frac{4}{3}\Delta$

$\Delta = 1$

Si y_3 coincide con x_4

Le INTEGRAL NUMERICA

$$= W_0 \frac{e^{-4^2}}{\sqrt{\pi}} + W_1 \frac{e^{-3^2}}{\sqrt{\pi}} + W_2 \frac{e^{-2^2}}{\sqrt{\pi}} + W_3 \frac{e^{-1^2}}{\sqrt{\pi}} + W_4 \frac{e^{-0^2}}{\sqrt{\pi}}$$

$$+ W_5 \frac{e^{-1^2}}{\sqrt{\pi}} + W_6 \frac{e^{-2^2}}{\sqrt{\pi}} + W_7 \frac{e^{-3^2}}{\sqrt{\pi}} + W_8 \frac{e^{-4^2}}{\sqrt{\pi}}$$

+ terminos despreciables

$$= \frac{3.169916}{3} = 1.056640$$

PEOR QUE CON TRAPECIOS

(pero es casualidad)

Si $\Delta = 1/2$

Le sume de 1.000034

Si $Y_j = x_q$ $\Delta = 1$

La integral vale 0.943567

MUCHISIMA DIFERENCIA

si $\Delta = 1/2$

la integral vale 0,999965

LUEGO HAY OSCILACIONES IMPORTANTES EN LOS VALORES DEPENDIENDO DE POSICIÓN DE Y_j:

" DEL VALOR DE LOS PESOS ASOCIADOS

ALTERNANCIA

$\frac{2}{3}\Delta$, $\frac{4}{3}\Delta$, $\frac{2}{3}\Delta$, $\frac{4}{3}\Delta$, etc

Y ESTO EN UN CASO MUY SIMPLE

MUCHOS,

MUCHÍSIMOS

DE LOS RESULTADOS

CATASTRÓFICOS EN LA

INVERSIÓN DE

TRANSFORMADAS

INTEGRALES

PROVIENEN DE

LA "DISCRETIZACIÓN"

DIRECTA CON

PESOS DE INTEGRACIÓN

"SIMPSON"

PERO...

REPETIDO DE INTEGRACIÓN NUMÉRICA

UN PROBLEMA MUCHO MÁS DELICADO APARECE CUANDO LA FUNCIÓN $F(x)$ DEPENDE DE UN PARÁMETRO λ: $F_\lambda(x)$ DE TAL FORMA QUE PUEDA CAMBIAR SU FORMA FUNCIONAL PARA DIFERENTES VALORES DE ESTE PARAMETRO

ENTONCES PUEDE OCURRIR QUE PARA CADA VALOR DE λ EXISTIRÁ UNA FORMA DE INTEGRACIÓN NUMÉRICA QUE OPTIMICE LA INTEGRAL

$$\int_A^B F_\lambda(x)\,dx = \sum_{j=1}^{N} W_\lambda(x_j)\, F_\lambda(x_j)$$

ES DECIR PARA CADA λ EL CONJUNTO DE ORDENADAS DISCRETAS Y EL CONJUNTO DE PESOS ASOCIADOS SERÁ DIFERENTE

Y, A VECES TENEMOS QUE HACER MUCHAS INTEGRALES PARA MUCHAS $F_\lambda(x)$ CON EL MISMO CONJUNTO DE ORDENADAS DISCRETAS $\{x_j\}$ Y EL MISMO CONJUNTO DE PESOS ASOCIADOS.

CONCLUSIÓN !!!

EN LAS ECUACIONES INTEGRALES SE PRESENTA, OBVIAMENTE, LA DIFICULTAD ANTERIOR

LA FUNCIÓN $F(x)$ QUE HAY QUE INTEGRAR, DEPENDE DEL PARÁMETRO y:

$$\int_A^B K(y,x)\, f(x)\, dx$$

ASÍ QUE: - MUCHO CUIDADO
- MUCHA EXPERIENCIA

O BUSCAR OTROS MÉTODOS.

EN EL CASO EN EL QUE EL INTERVALO DE INTEGRACIÓN (A,B) ES FINITO

LA REPRESENTACIÓN FUNCIONAL DEL INTEGRANDO SE BASA EN APROXIMACIONES POLINOMIALES DEL INTEGRANDO

Y UN POLINOMIO ES UN POLINOMIO AUNQUE SUS COEFICIENTES DEPENDAN DEL PARÁMETRO y.

ASÍ QUE, AL MENOS EN ESTE CASO PUEDE SERVIRNOS ESTE MÉTODO

EN EL CASO DE LA CONVOLUCIÓN GAUSSIANA

$$g(y) = \int_{-\infty}^{+\infty} e^{-(x-y)^2} f(x)\, dx$$

PARA CADA y TENEMOS UNA GAUSSIANA CENTRADA, PRECISAMENTE, EN ESE y

PODREMOS ELEGIR UN CONJUNTO COMUN DE ORDENADAS x_k

- DONDE EVALUAREMOS $f(x_k)$

Y EL CONJUNTO DE PESOS CORRESPONDIENTE w_k

PERO ESOS PESOS w_k ASOCIADOS AL CONJUNTO x_k TIENEN QUE SER TALES QUE

$$\int_{-\infty}^{+\infty} e^{-(x-y)^2} dx \simeq \sum_k e^{-(x_k-y)^2} w_k$$

SEA IGUAL PARA TODO y

LO QUE NO OCURRÍA ANTES

POR EJEMPLO

EN EL CASO DE LA TRANSFORMADA DE LAPLACE

$$g(y) = \int_0^\infty e^{-xy} f(x)\, dx$$

UN SISTEMA DE ORDENADAS DISCRETAS $\{x_k\}$ TALES QUE

$$g(y) = \sum_k w(k)\, e^{-x_k y} f(x_k)$$

SEA OPTIMA

DEPENDERA EXTRAORDINARIAMENTE DE y

Para cada y tendremos un conjunto optimo de valores x_k y w_k
Pero luego que resolver el problema con los valores de x_k y de w_k dos mismos para todos los y.